GOSFORD HIGH SCHOOL

2007
YEAR 12 HALF YEARLY HSC COURSE.

MATHEMATICS EXTENSION 1

General Instructions:

- Reading time – 5minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- Each question should be started on a new page.
- All necessary working should be shown in every question

Total marks: - 84

- Attempt Questions 1 -7
- All questions are of equal value

Question 1 (12 marks)

(i) Find the acute angle between the lines $2y-x+1=0$ and $5x-y+2=0$ correct to the nearest minute. (2)

(ii) Find the coordinates of the point P that divides the interval joining A (2,5) and B (-1,0) externally in the ratio 3:4. (2)

(iii) Solve $\frac{x+1}{x-1} \geq 3, x \neq 1$ and graph your solution on a number line. (3)

(iv) Differentiate with respect to x:

(a) $x^2 \cos^{-1} x$. (2)

(b) $\frac{\tan^{-1} x}{x}$. (3)

Question 2 (12marks)

(i) Find in simplest surd form the exact value of sin 75^0. (3)

(ii) If $\alpha, \beta \,\&\, \gamma$ are the roots of the equation $2x^3+5x-3=0$ find the value of

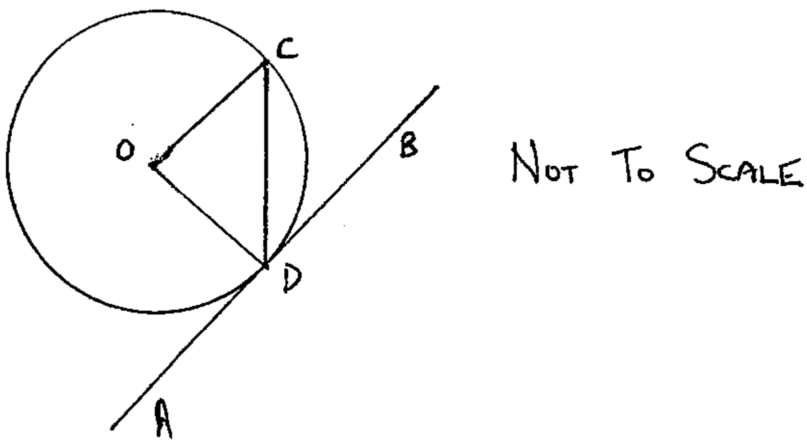
(a) $\alpha+\beta+\gamma$ (1)

(b) $\alpha\beta\gamma$ (1)

(c) $\alpha^{-1}+\beta^{-1}+\gamma^{-1}$ (2)

(iii) Draw a neat sketch of $y=2\sin^{-1}\frac{x}{3}$ and state its domain and range. (3)

(iii) AB is a tangent at D to the circle center O.
Calculate the size of $\angle$CDB given that $\angle$COD = 110^0 (2)



Question 3 (12 marks)

(i) Find

(a) $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ (2)

(b) $\displaystyle\int \sin^2 x\,dx$ (2)

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2 2x\,dx$ (2)

(d) $\displaystyle\int_1^2 2x\sqrt{x^2-1}\,dx$ using the substitution $u=x^2-1$ (3)

(ii) Use the substitution $u=\sin x$ to show that:

$$\int_0^{\frac{\pi}{6}} \frac{\cos x\,dx}{4\sin^2 x+1} = \frac{\pi}{8}$$ (3)

Question 4 (12 marks)

(i) Express $\tan 2A$ in terms of $\tan A$ and without using a calculator show that $\tan^{-1}\left(\frac{12}{5}\right)=2\tan^{-1}\left(\frac{2}{3}\right)$. (3)

(ii) (a) Write $\sqrt{3}\sin x-\cos x$ in the form $R\sin(x-\alpha)$ where $R\rangle 0\ \&\ 0\le\alpha\le\frac{\pi}{2}$. (1)

(b) Hence or otherwise solve $\sqrt{3}\sin x-\cos x=1$ for $0\le x\le 2\pi$ (2)

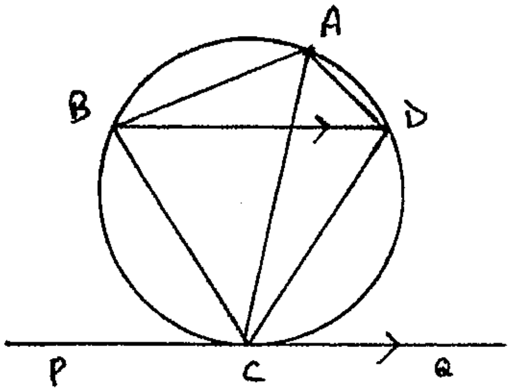
(iii) Prove by the principle of mathematical induction that:

$$1+4+4^2+\ldots+4^{n-1} = \frac{1}{3}\left(4^n-1\right)$$ (4)

(iv) If $y=\sin^{-1}\sqrt{x}$, show that $\frac{dy}{dx}=\frac{1}{\sin 2y}$ (2)

Question 5 (12 marks)

(i) Copy the following diagram onto your answer sheet.



Given that BD is parallel to PQ and PQ is a tangent to the circle at C, prove that CA bisects $\angle$BAD. (4)

(ii) (a) Show that the equation $\log_e x - \cos x = 0$ has a root between $x=1$ and $x=2$. (1)

(b) Using one application of Newton's Method find a value of the root of the equation $\log_e x - \cos x = 0$, which is close to $x=1.2$. (3)

(iii) The normal to the parabola $x^2 = 4ay$ at the point $P\left(2ap, ap^2\right)$ cuts the axis of the parabola at Q. If S is the focus of the parabola, find the area of triangle PQS. (4)

Question 6 (12 marks)

(i) Find all solutions to $\sin 2\theta = \cos\theta$ (3)

(ii) (a) Show that $\frac{d}{dx}\left[x\sin^{-1}x + \sqrt{1-x^2}\right] = \sin^{-1}x$ (2)

(b) Hence show that the volume of the solid of revolution formed by rotating the area under the curve $y=\sqrt{\sin^{-1}x}$ between $x=0$ & $x=\frac{1}{2}$, about the x-axis, is $\frac{\pi^2}{12} + \left(\sqrt{3}-2\right)\frac{\pi}{2}$ units^3. (3)

(iii) (a) On the same set of axes sketch the graphs of $y=2|x|$ & $y=|x+3|$ (2)

(b) Solve $2|x| \le |x+3|$ (2)

Question 7 (12 marks)

(i) Find $\tan\left[\sin^{-1}\frac{5}{13}-\sin^{-1}\frac{4}{5}\right]$ (2)

(ii) $P\left(4t,2t^2\right)$ and $Q\left(8t,8t^2\right)$ are two variable points on the parabola $x^2=8y$. The tangents from P and Q intersect at T. Find the Cartesian equation of the locus of T. (4)

(iii) (a) Sketch the graph of the function $f(x)=e^x-4$ clearly showing the coordinates of any points of intersection with the axes and the equations of any asymptotes. (2)

(b) On the same diagram sketch the graph of the function $y=f^{-1}(x)$ showing all important features. (1)

(c) Find an expression for $y=f^{-1}(x)$ in terms of x. (2)

(d) Explain why the x coordinates of any points of intersection of the two functions satisfy the equation $e^x-x-4=0$. (1)

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1.

(i) $M_1 = \frac{1}{2}$, $M_2 = 5$

$$\tan\theta = \left|\frac{M_1 - M_2}{1 + M_1 M_2}\right|$$

$$= \left|\frac{\frac{1}{2} - 5}{1 + \frac{1}{2}\times 5}\right|$$

$$= \frac{4\frac{1}{2}}{3\frac{1}{2}}$$

(2)

$\therefore \theta = 52^\circ 8'$ (nearest min)

(ii) $(2,5)$ $(-1,0)$

$3 : -4$

$$x = \frac{-4\times 2 + 3\times -1}{3 + -4}, \quad y = \frac{-4\times 5 + 3\times 0}{3 + -4}$$

$$= 11 \qquad\qquad = 20$$

(2)

P is the pt $(11, 20)$

(iii) $\frac{x+1}{x-1} > 3$

$x \neq 1$

Consider the eq^n :

$$\frac{x+1}{x-1} = 3$$

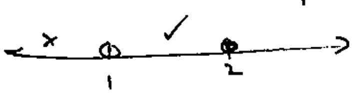
$$x + 1 = 3x - 3$$

$$4 = 2x$$

$$x = 2$$

Since this is the sol^n to the eq^n it cant be a sol^n to the ineq^n.

(3)

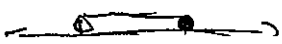


Test 0 : $\frac{1}{-1} \not> 3$

Test $1\frac{1}{2}$: $\frac{2\frac{1}{2}}{\frac{1}{2}} > 3$

Test 3 : $\frac{4}{2} \not> 3$.

$\therefore$ Sol^n is $1 < x \leq 2$



(iv) a) Let $y = x^2 \cos^{-1} x$

(2)

$$\frac{dy}{dx} = SF' + FS'$$

$$= \cos^{-1}x \cdot 2x + x^2 \cdot \frac{-1}{\sqrt{1-x^2}}$$

$$= 2x\cos^{-1}x - \frac{x^2}{\sqrt{1-x^2}}.$$

b) Let $y = \frac{\tan^{-1}x}{x}$

(3)

$$\frac{dy}{dx} = \frac{BT' - TB'}{B^2}$$

$$= \frac{x \cdot \frac{1}{1+x^2} - \tan^{-1}x \cdot 1}{x^2}$$

$$= \frac{\frac{x}{1+x^2} - \tan^{-1}x}{x^2}$$

$$= \frac{x - (1+x^2)\tan^{-1}x}{x^2(1+x^2)}$$

Question 2

(i) $\sin 75^\circ = \sin(45^\circ + 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$ 3

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

(ii) a) $\alpha+\beta+\gamma = -\frac{b}{a}$ 1

$$= 0$$

b) $\alpha\beta\gamma = -\frac{d}{a}$ 1

$$= \frac{3}{2}$$

c) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$$= \frac{\frac{c}{a}}{\frac{3}{2}}$$

$$= \frac{5/2}{3/2}$$ 2

$$= 5/3$$

(iii) If $y = 2\sin^{-1}\frac{x}{3}$

$$\frac{y}{2} = \sin^{-1}\frac{x}{3}$$

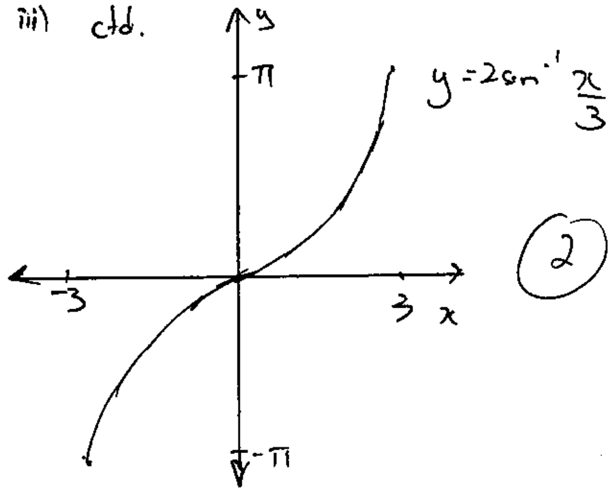
$\therefore R: -\frac{\pi}{2} \le \frac{y}{2} \le \frac{\pi}{2}$

ie $-\pi \le y \le \pi$ 1

D: $-1 \le \frac{x}{3} \le 1$

$$-3 \le x \le 3$$

iii) ctd.



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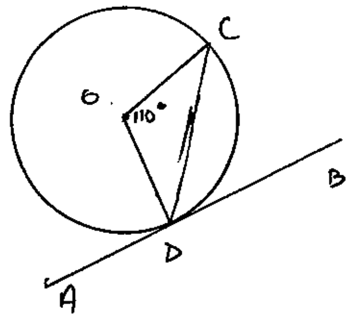
(iv)



$\angle OCD = \angle ODC = 35^\circ$ ($\angle$ sum of a Δ is 180° & base $\angle$'s of an isosc Δ) 2

$\therefore \angle CDB = 35^\circ$ ($\angle$ in the alt segment theorem).

Question 3 :

(i) a) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$

$= \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$ (2)

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1} 0$

$= \frac{\pi}{3} - 0$

$= \frac{\pi}{3}$.

b) $\int \sin^2 x \, dx$

$= \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$ (2)

$= \frac{x}{2} - \frac{1}{4}\sin 2x + c$

c) $\int_0^{\pi/2} \cos^2 2x \, dx$

$= \int_0^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos 2x \, dx$

$= \left[\frac{x}{2} + \frac{1}{4}\sin 2x\right]_0^{\pi/2}$ (2)

$= \left(\frac{\pi}{4} + \frac{1}{4}\sin\pi\right) - \left(\frac{0}{2} + \frac{1}{4}\sin 0\right)$

$= \left(\frac{\pi}{4} + 0\right) - (0 + 0)$

$= \frac{\pi}{4}$.

d) If $u = x^2 - 1$

$du = 2x \, dx$

When $x = 1$, $u = 0$

$x = 2$, $u = 3$.

d) ctd.

$\therefore I = \int_0^3 u^{1/2} \, du$

$= \left[\frac{2}{3} u^{3/2}\right]_0^3$ (3)

$= \frac{2}{3} \times 3^{3/2} - \frac{2}{3} \times 0$

$= \frac{2}{3} \times 3\sqrt{3}$

$= 2\sqrt{3}$.

(ii) If $u = \sin x$

$du = \cos x \, dx$

When $x = 0$, $u = 0$

$x = \frac{\pi}{6}$, $u = \frac{1}{2}$

$\therefore I = \int_0^{1/2} \frac{du}{4u^2+1}$

$= \frac{1}{4}\int_0^{1/2} \frac{du}{u^2 + \frac{1}{4}}$ (3)

$= \frac{1}{4} \times 2 \left[\tan^{-1} 2u\right]_0^{1/2}$

$= \frac{1}{2}\left\{\tan^{-1} 1 - \tan^{-1} 0\right\}$

$= \frac{1}{2} \times \left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi}{8}$.

Question 4:

(i) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Let $A = \tan^{-1}\left(\frac{2}{3}\right)$

$\therefore \tan A = \frac{2}{3}$

$\tan 2A = \dfrac{2 \times \frac{2}{3}}{1-\left(\frac{2}{3}\right)^2}$ (3)

$= \dfrac{\frac{4}{3}}{\frac{5}{9}}$

$= \frac{4}{3} \times \frac{9}{5}$

$= \frac{12}{5}.$

$\therefore 2A = \tan^{-1}\left(\frac{12}{5}\right)$

ie. $2\tan^{-1}\left(\frac{2}{3}\right) = \tan^{-1}\left(\frac{12}{5}\right)$

(ii) a $R = \sqrt{(\sqrt{3})^2 + 1^2}$

$= 2$

& $\tan\alpha = \frac{1}{\sqrt{3}}$ (1)

$\therefore \alpha = \frac{\pi}{6}.$

$\therefore \sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$

b) $\sqrt{3}\sin x - \cos x = 1$

$2\sin\left(x - \frac{\pi}{6}\right) = 1$

$\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$ (2)

$\therefore x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$

$x = \frac{\pi}{3}, \pi.$

(iii) Assume true for $n=k$

ie. $1+4+4^2+\dots+4^{k-1} = \frac{1}{3}(4^k-1)$

Prove true for $n=1$

LHS $= 1$

RHS $= \frac{1}{3}(4^1-1)$

$= 1$

$\therefore$ true for $n=1$.

Prove true for $n=k$

ie $1+4+\dots+4^{k-1}+4^k = \frac{1}{3}(4^{k+1}-1)$

LHS $= \frac{1}{3}(4^k-1) + 4^k$

$= \dfrac{(4^k-1)+3\times 4^k}{3}$ (4)

$= \dfrac{4\times 4^k - 1}{3}$

$= \frac{1}{3}(4^{k+1}-1)$

$\therefore$ If the statement is true for $n=k$ it is true for $n=k+1$. Since it is true for $n=1$ it is true for $n=2$ & so on $\therefore$ the statement is true for all positive integers n.

(iv) If $y = \sin^{-1}\sqrt{x}$

$\sqrt{x} = \sin y$

$x = \sin^2 y.$

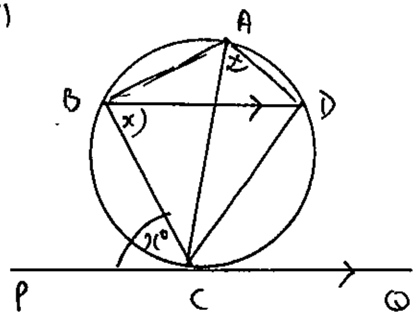
$\therefore \frac{dx}{dy} = 2\sin y\cos y$ (2)

$= \sin 2y$

$\therefore \frac{dy}{dx} = \frac{1}{\sin 2y}$

Question 5

(i)



Let $\angle PCB$ be $x°$

$\therefore$ $\angle CBD = x°$ (alt. $\angle$'s in parallel lines, $BD \parallel PQ$)

$\therefore$ $\angle DAC = x°$ ($\angle$'s in the same segment are equal)

4

Also $\angle BAC = x°$ ($\angle$ in the alt. segment theorem)

$\therefore$ $\angle DAC = \angle BAC$

ie. CA bisects $\angle BAD$.

(ii) a) $\log_e 1 - \cos 1 \doteqdot -0.54$ (2 d.p)

$\log_e 2 - \cos 2 \doteqdot 1.11$ (2 d.p) 1

$\therefore$ there is a root between 1 & 2

b) If $f(x) = \log_e x - \cos x$

$$f'(x) = \frac{1}{x} + \sin x$$

Now $a_2 = a_1 - \dfrac{f(a_1)}{f'(a_1)}$ 3

$$\doteqdot 1.2 - \left[\frac{-0.180036197}{1.765372419}\right]$$

$$\doteqdot 1.30 \text{ (2 d.p)}$$

(iii) If $x^2 = 4ay$, $y = \dfrac{x^2}{4a}$

$$\therefore \frac{dy}{dx} = \frac{2x}{4a}$$

When $x = 2ap$

$$\frac{dy}{dx} = \frac{4ap}{4a} = p$$

$\therefore$ The grad of the normal is $-\frac{1}{p}$

& the eq^n is

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$py - ap^3 = -x + 2ap$$

$$\text{ie } x + py = 2ap + ap^3.$$

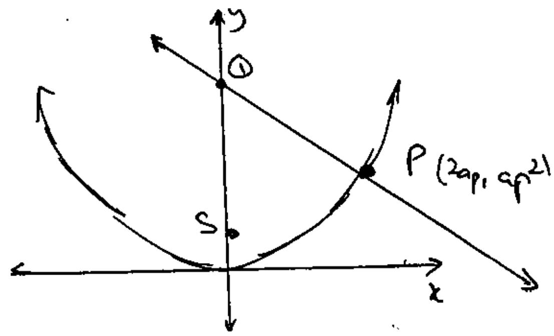
4

When $x = 0$, $py = 2ap + ap^3$

$$y = 2a + ap^2.$$

$\therefore$ Q is the pt $(0, 2a + ap^2)$

& S is the pt $(0, a)$



$\therefore$ $QS = 2a + ap^2 - a$

$= a + ap^2$

& Area $= \frac{1}{2} \times b \times h$

$= \frac{1}{2}(a + ap^2) \times 2ap$

$= ap(a + ap^2)$

$= a^2p + a^2p^3$ units^3

Question 6:

(i) $\sin 2\theta = \cos\theta$

$\sin 2\theta - \cos\theta = 0$

$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\therefore \cos\theta = 0 \quad \text{or} \quad \sin\theta = \frac{1}{2}$ (3)

$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

or $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots$

ie $\theta = 2n\pi \pm \frac{\pi}{2}$ or $n\pi + (-1)^n \cdot \frac{\pi}{6}$.

(ii) a) $\frac{d}{dx}\left[x\sin^{-1}x + \sqrt{1-x^2}\right]$

$= \frac{d}{dx}\left[x\sin^{-1}x + (1-x^2)^{\frac{1}{2}}\right]$ (2)

$= \sin^{-1}x \cdot 1 + x \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x$

$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$

$= \sin^{-1}x$.

b) $V = \pi\int_a^b y^2\,dx$ (3)

$= \pi\int_0^{\frac{1}{2}} \sin^{-1}x\,dx$

$= \pi\left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$

$= \pi\left[\left(\frac{1}{2}\sin^{-1}\frac{1}{2} + \sqrt{1-\frac{1}{4}}\right) - \left(0\cdot\sin^{-1}0 + \sqrt{1-0}\right)\right]$

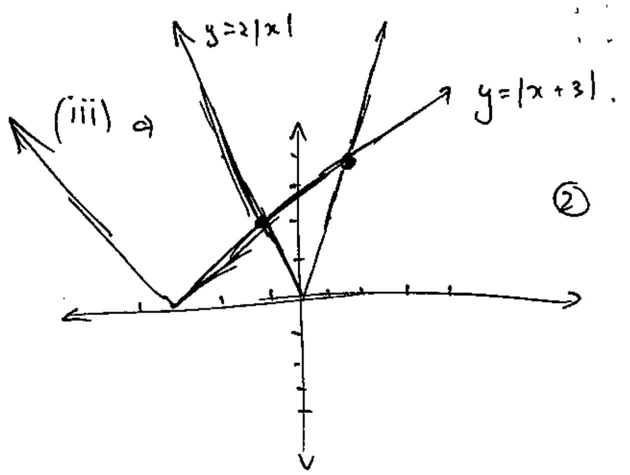
$= \pi\left[\left(\frac{1}{2}\cdot\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right) - (0+1)\right]$

$= \pi\left(\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1\right)$

$= \frac{\pi^2}{12} + \pi\left(\frac{\sqrt{3}-2}{2}\right)$

$= \frac{\pi^2}{12} + \frac{\pi}{2}(\sqrt{3}-2)$ units^3

(iii) a)



(2)

b) If $2x = x+3$, If $-2x = x+3$

$x = 3$ $\quad\quad -3x = 3$

$x = -1$

$\therefore$ From the graph if

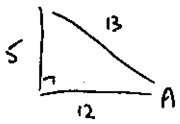
$2|x| \leq |x+3|$

the sol^n is: $-1 \leq x \leq 3$.

Question 7:

(i) Let $\sin^{-1}\frac{5}{13} = A$

$\therefore \sin A = \frac{5}{13}$



Let $\sin^{-1}\frac{4}{5} = B$

$\therefore \sin B = \frac{4}{5}$

$\therefore \tan\left[\sin^{-1}\frac{5}{13} - \sin^{-1}\frac{4}{5}\right]$

$= \tan[A - B]$

$= \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$= \dfrac{\frac{5}{12} - \frac{4}{3}}{1 + \frac{5}{12} \times \frac{4}{3}}$ (3)

$= \dfrac{\frac{5}{12} - \frac{16}{12}}{1 + \frac{5}{9}}$

$= \dfrac{-11/12}{14/9}$

$= \dfrac{-11}{\cancel{12}_{4}} \times \dfrac{\cancel{9}^{3}}{14}$

$= \dfrac{-33}{56}$.

(ii) If $x^2 = 8y$

$y = \frac{x^2}{8}$

$\frac{dy}{dx} = \frac{2x}{8}$

If $x = 4t$, $\frac{dy}{dx} = t$

If $x = 8t$, $\frac{dy}{dx} = 2t$.

$\therefore$ Eq^n of tangent at P is

$y - 2t^2 = t(x - 4t)$

$y - 2t^2 = tx - 4t^2$

i.e. $y = tx - 2t^2$ --- (1)

& Eq^n of tangent at Q is

$y - 8t^2 = 2t(x - 8t)$

$y - 8t^2 = 2tx - 16t^2$

i.e. $y = 2tx - 8t^2$ --- (2)

$\therefore$ T is given by the pt of intersection of (1) & (2)

$\therefore tx - 2t^2 = 2tx - 8t^2$

$6t^2 = tx$

$x = 6t$

$y = 4t^2$ (4)

T is the pt $(6t, 4t^2)$

If $x = 6t$

$t = \frac{x}{6}$

& if $y = 4t^2$

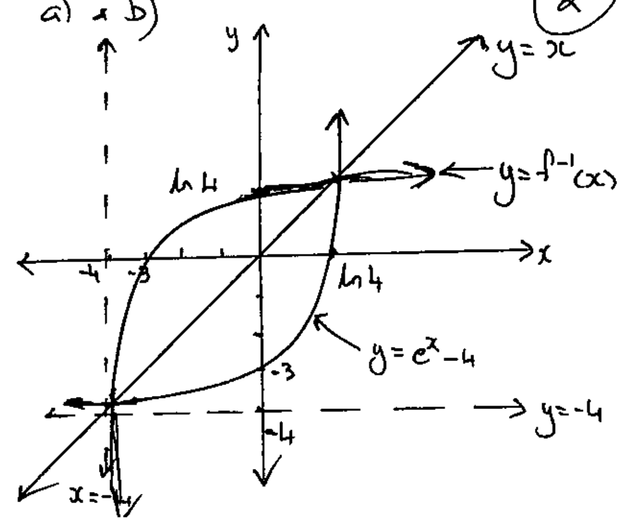
$y = 4 \times \left(\frac{x}{6}\right)^2$

$= \frac{4x^2}{36}$

i.e. $y = \frac{x^2}{9}$

(iii)

a) & b) (2)



c) $f: \quad y = e^x - 4$

is a one to one function

$\therefore$ its inverse exists

$f^{-1}: \quad x = e^y - 4$

$\therefore e^y = x + 4$ (1)

$y = \log_e(x+4)$

d) The point of intersection of the two curves must lie on the line $y = x$ since they are inverses (2)

$\therefore e^x - 4 = x$

ie $e^x - 4 - x = 0$.