GOSFORD HIGH SCHOOL

SPECTEMUR AGENDO

Year 12

2008

Higher School Certificate

MATHEMATICS EXTENSION I

Half Yearly Examination

Time Allowed: 2 Hours + 5 minutes reading time

Instructions:

- Start each question in a new booklet (extra booklets are available).
- Attempt questions 1-7.
- All questions are of equal value.
- Board approved calculators may be used.
- Write using black or blue pen.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question.

QUESTION 1: (12 Marks) Use a separate writing booklet.

Marks

a. Expand and simplify

$(3x-5)(2x+1)-(2x+3)^2$ 2

b. Factorise completely

$$x^2z+xyz+y^2z-x^3+y^3$$ 2

c. Differentiate with respect to x

$$(5x+1)^3(x-9)^5$$ 2

Expressing your answer in a form that is fully factorised.

d. Divide $P(x)=3x^5-7x^3+8x^2-5$ by $x-2$ and write $P(x)$ in the form $P(x)=(x-2)Q(x)+R(x)$ 3

e. If α,β,γ are the roots of

$$x^3-2x^2+3x-5=0$$ 3

Evaluate:

i. $\alpha+\beta+\gamma$

ii. $\alpha\beta\gamma$

iii. $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$

QUESTION 2: (12 Marks) Use a separate writing booklet.

Marks

a. Differentiate $\sin^3 x$ with respect to x. 1

b. Find $\int 4\sec^2(\frac{x}{2})dx$ 1

c. Solve $2\cos^2 x - \sqrt{3}\cos x = 0$
for $0 \le x \le 2\pi$, writing your answers in exact radian form. 2

d. i. Express $6\sin x + 8\cos x$ in the form $A\sin(x+\theta)$ where $A > 0$, $0 \le \theta \le \frac{\pi}{2}$ 1

ii. Hence, solve $6\sin x + 8\cos x = 5$ for $0 \le x \le 2\pi$ 2

(Answers in radians correct to 2 decimal places)

e. Show that $\dfrac{\cos\theta}{\sin\theta + 1} = \dfrac{1-t}{1+t}$ 2
where $t = \tan\dfrac{\theta}{2}$

f. The area under the curve $y = \sin x + \cos x$ above the x axis and between $x = 0$ and $x = \dfrac{\pi}{2}$ is rotated about the x axis. Find the volume of the solid of revolution formed. *(Leave answer in exact form)* 3

QUESTION 3: (12 Marks) Use a separate writing booklet.

Marks

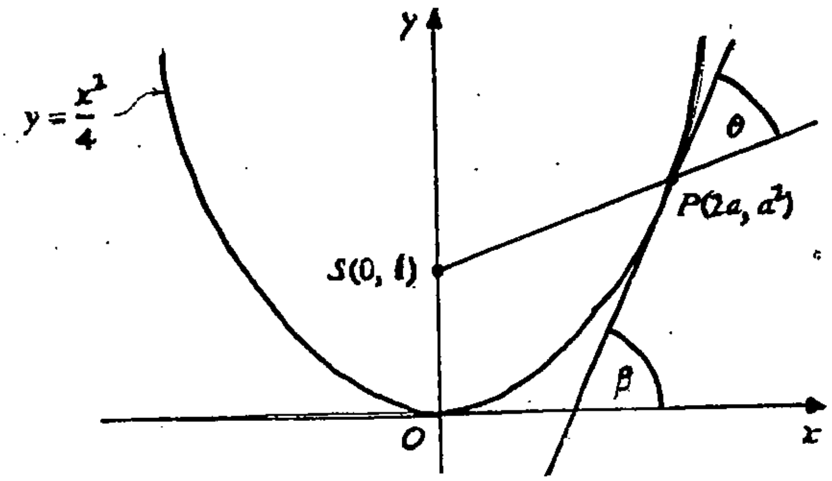
a. Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$

i. Show that the equation of the tangent to parabola at P is $y = px - ap^2$ 1

ii. The tangent at P and the line through Q parallel to the y axis intersect at T. Find the co-ordinates of T. 1

iii. Write down the co-ordinates of M, the midpoint of PT 1

iv. Describe the locus of M when $pq = -1$ 1



Let $P(2a, a^2)$ be a point on the parabola $y = \frac{x^2}{4}$ and let S be the point $(0,1)$.

The tangent to the parabola at P makes an angle of β with the x axis. The angle between SP and the tangent is θ.

Assume that $a > 0$.

Show that

i. $\tan\beta = a$ 1

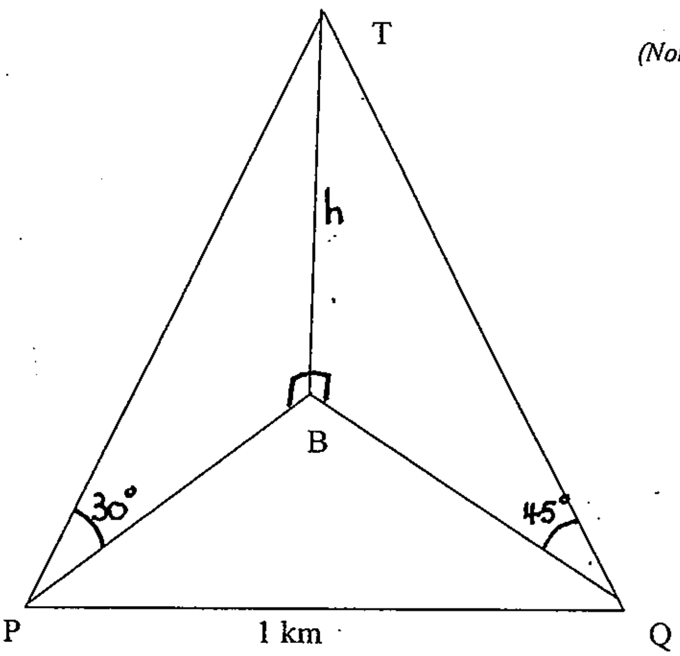
ii. the gradient of SP is $\frac{1}{2}(a - \frac{1}{a})$ 1

iii. $\tan\theta = \frac{1}{a}$ 2

Question 3 *continued next page*

Question 3 *(Continued)*

c.



The angle of elevation from a boat P to a point T at the top of a vertical cliff is measured to be 30°. The boat sails 1 km to a second point Q, from which the angle of elevation of T is 45°. Let B be the point at the base of the cliff directly below T and let h = BT, the height of the cliff in metres. The bearings of B from P and Q are 50° and 310° respectively.

i. Show that $\angle PBQ = 100°$ 1

ii. Find expression for PB and QB in terms of h 1

iii. Hence, show that 1

$$h^2 = \frac{1000^2}{\cot^2 30° + \cot^2 45° - 2\cot 30° \cot 45° \cos 100°}$$

iv. Use a calculator to find the height of the cliff, correct to the nearest metre. 1

QUESTION 4: (12 Marks) Use a separate writing booklet.

Marks

a. Given that

$$\frac{1}{1\times 4}+\frac{1}{4\times 7}+........+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1}$$ 2

prove that

$$\frac{1}{1\times 4}+\frac{1}{4\times 7}+........+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}=\frac{k+1}{3k+4}$$

b. Use the principle of Mathematical Induction to prove that

$2^{3n}-3^{n}$ is divisible by 5 4

for all positive integers n

c. i. Prove that $f(x) = x^3+3x-18$ has a root in the interval $2 < x < 2.5$ 1

ii. Use one application of the 'halving the interval' method to find a smaller interval containing the root. 1

iii. Which end of the small interval found in (ii) is closer to the root. Justify your answer. 1

d. Given that 9.5 is an approximate root of $x^3-6x^2+25x-500=0$, find by Newton's Method a better approximation. (Give your answer to 2 decimal places.) 3

QUESTION 5: (12 Marks) Use a separate writing booklet.

Marks

a. Find:

i. $\int \frac{\cos x}{\sin x} dx$ — 1

ii. $\int \sin^2 4x dx$ — 2

b. Use the table of standard integrals to evaluate:

$$\int_2^3 \frac{dx}{\sqrt{x^2-4}}$$ — 2

c. Evaluate:

$$\int_0^{\pi} (1+\cos x)^2 dx$$ — 2

d. Find:

$$\int \frac{x}{\sqrt{1+x^2}} \quad dx$$ — 2

Using the substitution $x = \tan\theta$

e. Evaluate

$$\int_0^3 \frac{3x}{\sqrt{1+x}} dx$$ — 3

Using the substitution $u = 1 + x$

QUESTION 6: (12 Marks) Use a separate writing booklet.

Marks

a. A tangent is drawn to $y = 2\tan^{-1} x$ at point P where $x = 1$

i. Find the y co-ordinate of P. 1

ii. Find the gradient of the normal at P. 1

b. Evaluate

$$\int_0^{\sqrt{3}} \frac{4}{9+x^2}\,dx$$ 2

c. Determine the exact value of

$$\cos\left(\sin^{-1}\left(\frac{-12}{13}\right)\right)$$ 2

d. For $f(x) = x^2 - 6x + 6$

i. Find the co-ordinates of the vertex. 1

ii. Explain why *f(x)* does not have an inverse function and state the restricted domain of *f(x)* over which an inverse exists which contains all positive x values. Call this inverse function $y = f^{-1}(x)$ 1

iii. State the domain of $f^{-1}(x)$ 1

iv. Find an expression for $y = f^{-1}(x)$ in terms of x. 2

v. Find the x values for the points of intersection of $y = f(x)$ and $y = f^{-1}(x)$ 1

QUESTION 7: (12 Marks) Use a separate writing booklet.

Marks

a. Solve $\dfrac{|x-1|-2}{12+x-x^2} \geq 0$ 2

b. A function $y = f(x)$ is defined by

$$f(x) = 2\cos^{-1}\left(\frac{x}{\sqrt{2}}\right) - \sin^{-1}\left(1-x^2\right)$$

and restricted to values of $x > 0$

i. Find the domain of $f(x)$ 1

ii. Find $f'(x)$ 2

iii. Hence, sketch $y = f(x)$ 1

c. Prove that:

$$\frac{\sin\left(\dfrac{3x}{2}\right)}{2\sin\left(\dfrac{x}{2}\right)} = \frac{1}{2} + \cos x$$ 3

d. Given that the equation

$P(x) = x^4 - 4kx^2 + 12$ 3

has a double root, find all possible value(s) of k

END OF TEST

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

a) $(3x-5)(2x+1)-(2x+3)^2$

$= 6x^2+3x-10x-5-(4x^2+12x+9)$

$= 6x^2-7x-5-4x^2-12x-9$

$= 2x^2-19x-14$

b) $x^2z+xyz+y^2z-x^3+y^3$

$=(x^2z+xyz+y^2z)-(x^3-y^3)$

$= z(x^2+xy+y^2)-(x-y)(x^2+xy+y^2)$

$= (x^2+xy+y^2)(z-x+y)$

c) Let $y=(5x+1)^3(x-9)^5$

$y' = (x-9)^5 . 15(5x+1)^2 + (5x+1)^3 . 5(x-9)^4$

$= 5(5x+1)^2(x-9)^4\left[(x-9).3+(5x+1)\right]$

$= 5(5x+1)^2(x-9)^4\left[3x-27+5x+1\right]$

$= 5(5x+1)^2(x-9)^4(8x-26)$

$= 10(5x+1)^2(x-9)^4(4x-13)$

d)

$$\begin{array}{r|l}
 & 3x^4+6x^3+5x^2+18x+36 \\
x-2 & 3x^5 \quad -7x^3+8x^2 \quad -5 \\
 & 3x^5-6x^4 \\
 & 6x^4-7x^3 \\
 & 6x^4-12x^3 \\
 & 5x^3+8x^2 \\
 & 5x^3-10x^2 \\
 & 18x^2 \quad -5 \\
 & 18x^2-36x \\
 & 36x-5 \\
 & 36x-72 \\
 & 67
\end{array}$$

$\therefore P(x) = (x-2)(3x^4+6x^3+5x^2+18x+36) + 67.$

e) $\alpha+\beta+\gamma = -\frac{b}{a} = 2$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} = 3$

$\alpha\beta\gamma = -\frac{d}{a} = 5$

(i) 2

(ii) 5

(iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{3}{5}$

2 a) Let $y = \sin^3 x$

$y' = 3\sin^2 x \cdot \cos x$

$= 3\cos x \sin^2 x$

b) $\int 4\sec^2\left(\frac{x}{2}\right) dx$

$= 2 \cdot 4 \tan\left(\frac{x}{2}\right) + c$

$= 8\tan\left(\frac{x}{2}\right) + c$

c) $2\cos^2 x - \sqrt{3}\cos x = 0$

$\cos x\left(2\cos x - \sqrt{3}\right) = 0$

$\therefore \cos x = 0, \quad 2\cos x - \sqrt{3} = 0$

$\cos x = \frac{\sqrt{3}}{2}$

$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{11\pi}{6}$

$\therefore x = \frac{\pi}{\ }, \frac{\pi}{\ }, \frac{3\pi}{\ }, \frac{11\pi}{\ }$

d) (i) $6\sin x + 8\cos x = A\sin(x+\theta)$

$A = \sqrt{6^2+8^2}$ $\quad \tan\theta = \frac{8}{6}$

$= 10$

$\therefore \theta = \tan^{-1}\frac{4}{3}$

$\therefore 6\sin x + 8\cos x = 10\sin\left(x + \tan^{-1}\left(\frac{4}{3}\right)\right)$

(ii) $10\sin\left(x + \tan^{-1}\frac{4}{3}\right) = 5$

$\sin\left(x + \tan^{-1}\frac{4}{3}\right) = \frac{1}{2}$

$\therefore x + \tan^{-1}\frac{4}{3} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots$

$\therefore x + 0.927 = \cancel{\frac{\pi}{6}}, \frac{5\pi}{6}, \frac{13\pi}{6}$

$x = 1.69, \ 5.88$ (2d.p)

e) $\dfrac{\cos\theta}{\sin\theta + 1} = \dfrac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2} + 1}$

$= \dfrac{\frac{1-t^2}{\cancel{1+t^2}}}{\frac{2t+1+t^2}{\cancel{1+t^2}}}$

$= \dfrac{1-t^2}{(1+t)^2}$

$= \dfrac{(1-t)(1+t)}{(1+t)^2}$

$= \dfrac{1-t}{1+t}$ as req.

f) $V = \pi\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 dx$

$= \pi\int_0^{\frac{\pi}{2}} \sin^2 x + 2\sin x\cos x + \cos^2 x$

$= \pi\int_0^{\frac{\pi}{2}} 1 + 2\sin x\cos x \, dx$

$$= \pi \int_0^{\frac{\pi}{2}} 1 + \sin 2x \, dx$$

$$= \pi \left[x - \frac{1}{2}\cos 2x \right]_0^{\frac{\pi}{2}}$$

$$= \pi \left[\left(\frac{\pi}{2} - \frac{1}{2}\cos\pi\right) - \left(0 - \frac{1}{2}\cos 0\right) \right]$$

$$= \pi \left[\left(\frac{\pi}{2} + \frac{1}{2}\right) - \left(-\frac{1}{2}\right) \right]$$

$$= \pi \left[\frac{\pi}{2} + 1 \right] \text{ units}^3$$

3/ a) (i) $x^2 = 4ay$

$$y = \frac{x^2}{4a}$$

$$y' = \frac{2x}{4a}$$

(when $x = 2ap$) $y' = \frac{2(2ap)}{4a}$

$$= p.$$

$\therefore$ Eqn of tangent $\Rightarrow$

$$y - ap^2 = p(x - 2ap)$$

$$y - ap^2 = px - 2ap^2$$

$$y = px - ap^2 \text{ as req.}$$

(ii) line through Q: $x = 2aq$

Solving $y = px - ap^2$ —(1)

$x = 2aq$ —(2)

Sub (2) in (1): $y = 2apq - ap^2$

$\therefore T = (2aq, 2apq - ap^2)$

(iii) $M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + 2apq - ap^2}{2} \right)$

$$= \left(a(p+q), apq \right)$$

(iv) when $pq = -1$

$$M = \left(a(p+q), -a \right)$$

Since the y co-ordinate is $-a$ the locus of m is the directrix

b) (i) $y' = \frac{2x}{4} = \frac{x}{2}$

(at $x = 2a$) $y' = \frac{2a}{2} = a$

$\therefore$ gradient of tangent $= a$

$\therefore \tan\beta = a$

(since $\tan\beta = m$)

(ii) gradient SP $= \frac{a^2 - 1}{2a}$

$$= \frac{a}{2} - \frac{1}{2a}$$

$$= \frac{1}{2}\left(a - \frac{1}{a}\right)$$

as req.

(iii) $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$$\tan\theta = \frac{a - \frac{1}{2}\left(a - \frac{1}{a}\right)}{1 + \frac{a}{2}\left(a - \frac{1}{a}\right)}$$

$$\therefore \tan\theta = \frac{a - \frac{a}{2} + \frac{1}{2a}}{1 + \frac{a^2}{2} - \frac{1}{2}}$$

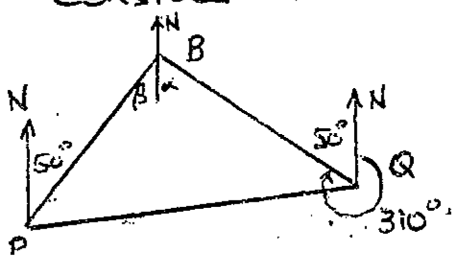
$$= \frac{\frac{a}{2} + \frac{1}{2a}}{\frac{a^2}{2} + \frac{1}{2}}$$

$$= \frac{a + \frac{1}{a}}{a^2 + 1}$$

$$= \frac{\frac{a^2+1}{a}}{a^2+1}$$

$$= \frac{1}{a} \quad \text{as req.}$$

c) (i) Consider :



From diagram : $\alpha = 50°$ (alternate ∠'s in ∥ are =)

Similarly $\beta = 50°$

$\therefore \hat{PBQ} = 100°$ (angle sum of Δ PBQ = 180°)

(ii) $\tan 30° = \frac{h}{PB}$

$$PB = \frac{h}{\tan 30°}$$

$$= h\cot 30°$$

$$\therefore QB = h\cot 45°$$

(iii) using Δ PBQ (cosine rule)

$$1000^2 = h^2\cot^2 30° + h^2\cot^2 45° - 2h^2\cot 30°\cot 45°.\cos 100°$$

$$1000^2 = h^2\left[\cot^2 30° + \cot^2 45° - 2\cot 30°\cot 45°.\cos 100°\right]$$

$$\therefore h^2 = \frac{1000^2}{\cot^2 30° + \cot^2 45° - 2\cot 30°\cot 45°\cos 100°}$$

as req.

iv) By calculation, the height of the cliff is 466 m (to nearest m.)

4/ a) LHS =

$$\frac{1}{1\times4} + \frac{1}{4\times7} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k(3k+4)+1}{(3k+1)(3k+4)}$$

$$= \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{\cancel{(3k+1)}(k+1)}{\cancel{(3k+1)}(3k+4)}$$

$$= \frac{k+1}{3k+4}$$

$= RHS$ as req.

b/ Step 1. To prove true for $n=1$

$2^{3n} - 3^n$ when $n=1$

$= 2^3 - 3$

$= 5$ which is divisible by 5

$\therefore$ true for $n=1$

Step 2. Assume true for $n=k$

i.e $\frac{2^{3k} - 3^k}{5} = M$

where M is a whole number

i.e $2^{3k} - 3^k = 5M$

$2^{3k} = 5M + 3^k$

Step 3 to prove true for $n=k+1$

i.e $2^{3(k+1)} - 3^{k+1}$ is divisible by 5.

$$2^{3(k+1)} - 3^{k+1} = 2^{3k+3} - 3^{k+1}$$
$$= 2^3 \times 2^{3k} - 3 \times 3^k$$
$$= 8.2^{3k} - 3.3^k$$
$$= 8(5M + 3^k) - 3.3^k$$
$$= 40M + 8.3^k - 3.3^k$$
$$= 40M + 5.3^k$$
$$= 5(8M + 3^k)$$

which is divisible by 5.

$\therefore$ if true for $n=k$ then true for $n=k+1$

Step 4 Since true for $n=1$ then it must be true for $n=1+1=2$ and if true for $n=2$, then it must be true for $n=2+1=3$ and so on for all positive integers n.

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c/ (i) $f(2) = 2^3 + 3\times2 - 18$

$= 8 + 6 - 18$

$= -4 < 0$

and $f(2.5) = 2.5^3 + 3\times2.5 - 18$

$= 15.625 + 7.5 - 18$

$= 5.125 > 0$

$\therefore$ a root exists in the interval

$2 < x < 2.5$

(ii) let $x = 2.25$

$f(2.25) = 2.25^3 + 3\times2.25 - 18$

$= 0.140625 > 0$

$\therefore$ a smaller interval is

$2 < x < 2.25$

(iii) Since $f(2.25)$ is closer to zero than $f(2)$ then the root is closer to 2.25

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d/ let $f(x) = x^3 - 6x^2 + 25x - 50$

$f'(x) = 3x^2 - 12x + 25$

(if $x_1 = 9.5$) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_2 = 9.5 - \frac{f(9.5)}{f'(9.5)}$

$f(9.5) = 53.375$
$f'(9.5) = 181.75$

$x_2 = 9.5 - \frac{53.375}{181.75}$

$= 9.21$ (2 d.p.)

5/ a) (i) $\displaystyle\int \frac{\cos x}{\sin x}\,dx$

$$= \ln(\sin x) + c$$

(ii) $\displaystyle\int \sin^2 4x\,dx$

$$= \int \tfrac{1}{2}(1-\cos 8x)\,dx$$

$$= \tfrac{1}{2}\left(x - \tfrac{1}{8}\sin 8x\right) + c$$

$$= \tfrac{1}{2}x - \tfrac{1}{16}\sin 8x + c$$

b) $\displaystyle\int_2^3 \frac{dx}{\sqrt{x^2-4}}$

$$= \left[\ln\left(x + \sqrt{x^2-4}\right)\right]_2^3$$

$$= \left[\ln(3+\sqrt{5}) - \ln(2+0)\right]$$

$$= \ln\left(\frac{3+\sqrt{5}}{2}\right)$$

c) $\displaystyle\int_0^{\pi} (1+\cos x)^2\,dx$

$$= \int_0^{\pi} 1 + 2\cos x + \cos^2 x\,dx$$

$$= \int_0^{\pi} 1 + 2\cos x + \tfrac{1}{2}(1+\cos 2x)\,dx$$

$$= \int_0^{\pi} \tfrac{3}{2} + 2\cos x + \tfrac{1}{2}\cos 2x\,dx$$

$$= \left[\tfrac{3}{2}x + 2\sin x + \tfrac{1}{4}\sin 2x\right]_0^{\pi}$$

$$= \left(\tfrac{3\pi}{2} + 2\sin\pi + \tfrac{1}{4}\sin 2\pi\right) - 0$$

$$= \frac{3\pi}{2}$$

d) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$ $\quad \left\{ x = \tan\theta,\ \frac{dx}{d\theta} = \sec^2\theta,\ dx = \sec^2\theta\,d\theta \right.$

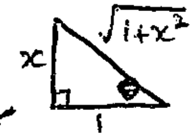
$$= \int \frac{\tan\theta\,.\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \int \frac{\tan\theta\,.\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta$$

$$= \int \frac{\tan\theta\sec^2\theta}{\sec\theta}\,d\theta$$

$$= \int \tan\theta\sec\theta\,d\theta$$

$$= \sec\theta + c$$



$$= \sqrt{1+x^2} + c$$

e) $\displaystyle\int_0^3 \frac{3x}{\sqrt{1+x}}\,dx$ $\quad \left\{ u = 1+x \Rightarrow x = u-1,\ \frac{du}{dx} = 1 \right.$

$du = dx$

$x = 3 \quad u = 4$

$x = 0 \quad u = 1$

$$= \int_1^4 \frac{3(u-1)}{\sqrt{u}}\,du$$

$$= \int_1^4 3u^{1/2} - 3u^{-1/2}\,du$$

$$= \left[\frac{3u^{3/2}}{3/2} - \frac{3u^{1/2}}{1/2}\right]_1^4$$

$$= \left[2u^{3/2} - 6u^{1/2}\right]_1^4$$

$$= (2\times 8 - 12) - (2 - 6)$$

$$= 8$$

6/ a) (i) $y = 2\tan^{-1}x$

$(x=1)$ $\quad y = 2\tan^{-1}1$

$$= \frac{\pi}{2}$$

(ii) $y' = \dfrac{2}{1+x^2}$

when $x=1$ $\quad y' = \frac{2}{2} = 1$

$\therefore$ gradient of normal $= -1$

eqn of normal $\Rightarrow$

$$y - \frac{\pi}{2} = -1(x-1)$$

$$y - \frac{\pi}{2} = -x+1$$

$$x + y - \frac{\pi}{2} - 1 = 0$$

b) $\displaystyle\int_0^{\sqrt{3}} \frac{4}{9+x^2}\,dx$

$$= \frac{4}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^{\sqrt{3}}$$

$$= \frac{4}{3}\left(\tan^{-1}\frac{\sqrt{3}}{3} - \tan^{-1}0\right)$$

$$= \frac{4}{3}\left(\frac{\pi}{6} - 0\right)$$

$$= \frac{2\pi}{9}$$

c) $\cos\left(\sin^{-1}\left(-\frac{12}{13}\right)\right)$

Since $\sin^{-1}$ is an odd function

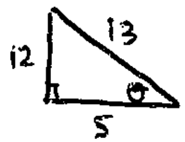
$$\sin^{-1}\left(-\frac{12}{13}\right) = -\sin^{-1}\left(\frac{12}{13}\right)$$

$$\therefore \cos\left(\sin^{-1}\left(-\frac{12}{13}\right)\right) = \cos\left(-\sin^{-1}\left(\frac{12}{13}\right)\right)$$

$$= \cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$$

since $\cos(-\theta) = \cos\theta$

Now



$\therefore \sin\theta = \frac{12}{13}$

$\theta = \sin^{-1}\left(\frac{12}{13}\right)$

$$\therefore \cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right) = \cos\theta$$

$$= \frac{5}{\ }$$

d) $f(x) = x^2 - 6x + 6$

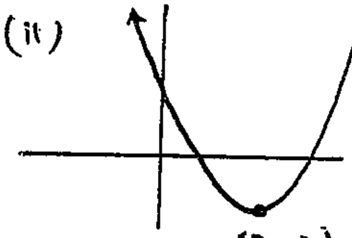
(i) $x = -\dfrac{b}{2a}$

$$= \frac{6}{2}$$

$x = 3 \Rightarrow$ Axis of symmetry

$\therefore$ Vertex $= (3, -3)$

(ii)



No inverse function since it doesn't pass the horizontal line test $\Rightarrow$ restricted domain $x \geqslant 3$

(iii) Domain of $f^{-1}(x) \Leftrightarrow$ Range of $f(x)$

$\therefore x \geqslant -3$

(iv) $x = y^2 - 6y + 6$

$$x - 6 = y^2 - 6y$$

$$x - 6 + 9 = y^2 - 6y + 9$$

$$x + 3 = (y-3)^2$$

$\therefore \ y - 3 = \pm\sqrt{x+3}$

$y = 3 \pm \sqrt{x+3}$

(v) Solve $y = x^2 - 6x + 6$ and $y = x$

$x = x^2 - 6x + 6$

$0 = x^2 - 7x + 6$

$0 = (x-6)(x-1)$

$\therefore \ x = 6, 1.$

7/a) $\dfrac{|x-1| - 2}{12 + x - x^2} \geq 0$

Consider $\dfrac{|x-1| - 2}{(4-x)(3+x)} = 0$

$\therefore \ x \neq 4, -3$

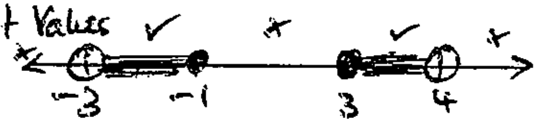
$|x-1| - 2 = 0$

$(x-1) = 2$

$x - 1 = 2 \ , \quad x - 1 = -2$

$x = 3 \qquad x = -1.$

Test Values ✓ × ✓ ×



$\therefore \ -3 < x \leq -1, \ 3 \leq x < 4$

b) $f(x) = 2\cos^{-1}\left(\dfrac{x}{\sqrt{2}}\right) - \sin^{-1}(1-x^2)$

(i) Domain:

$-1 \leq \dfrac{x}{\sqrt{2}} \leq 1 \ , \quad -1 \leq 1 - x^2 \leq 1$

$-\sqrt{2} \leq x \leq \sqrt{2} \qquad -2 \leq -x^2 \leq 0$

$0 \leq x^2 \leq 2$

$-\sqrt{2} \leq x \leq +\sqrt{2}$

$\therefore$ Domain: $0 < x \leq \sqrt{2}$

restriction.

(ii) $f'(x) = \dfrac{-2}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{1 - \frac{x^2}{2}}} - \dfrac{-2x}{\sqrt{1-(1-x^2)^2}}$

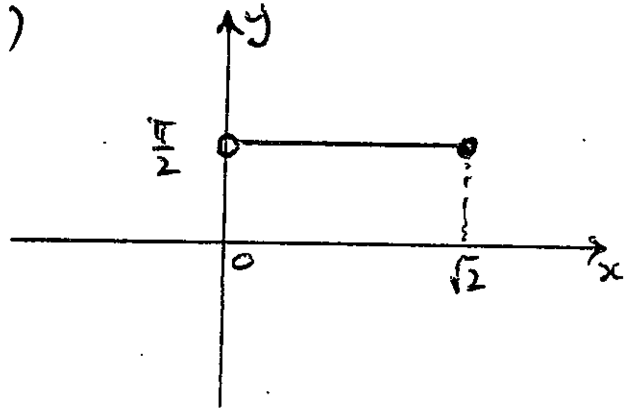
$= \dfrac{-2}{\sqrt{2-x^2}} + \dfrac{2x}{\sqrt{1 - 1 + 2x^2 - x^4}}$

$= \dfrac{-2}{\sqrt{2-x^2}} + \dfrac{2x}{\sqrt{2x^2 - x^4}}$

$= \dfrac{-2}{\sqrt{2-x^2}} + \dfrac{2x}{x\sqrt{2-x^2}}$

$= 0$

(iii)



Since $f'(x) = 0$ $f(x)$ is a constant

when $x = \sqrt{2}$ $y = \frac{\pi}{2}$

$\therefore \ y = \frac{\pi}{2}$ is the constant

c) $\dfrac{\sin\left(\frac{3x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$

$= \dfrac{\sin\left(x+\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$

$= \dfrac{\sin x.\cos\frac{x}{2} + \cos x.\sin\frac{x}{2}}{2\sin\frac{x}{2}}$

$= \dfrac{2\sin\frac{x}{2}.\cos\frac{x}{2}.\cos\frac{x}{2} + \cos x.\sin\frac{x}{2}}{2\sin\frac{x}{2}}$

$= \dfrac{\cancel{\sin\frac{x}{2}}\left(2\cos^2\frac{x}{2} + \cos x\right)}{2\cancel{\sin\frac{x}{2}}}$

$= \cos^2\frac{x}{2} + \frac{1}{2}\cos x$

$= \frac{1}{2}(1+\cos x) + \frac{1}{2}\cos x$

$= \frac{1}{2} + \frac{1}{2}\cos x + \frac{1}{2}\cos x$

$= \frac{1}{2} + \cos x$ as required

d) Let $P(x)$ have a double root at $x=\alpha$

$\therefore P(\alpha) \Rightarrow \alpha^4 - 4k\alpha^2 + 12 = 0$ — (1)

$P'(x) = 4x^3 - 8kx$

$\therefore P'(\alpha) \Rightarrow 4(\alpha)^3 - 8k\alpha = 0$ — (2)

Solving (2) : $4\alpha^3 - 8k\alpha = 0$

$4\alpha(\alpha^2 - 2k) = 0$

$\therefore \alpha = 0, \pm\sqrt{2k}$

Clearly $\alpha = 0$ is not a solution as $P(0) \neq 0$.

$P(\sqrt{2k}) = 4k^2 - 8k^2 + 12 = 0$

$-4k^2 + 12 = 0$

$k^2 = 3$

$\therefore k = \pm\sqrt{3}$

$P(-\sqrt{2k})$ gives the same values of k

$\therefore k = \pm\sqrt{3}$.