Student Name ....

Gosford High School

2009
Higher School Certificate

Mathematics – Extension 1
Half-Yearly Examination - 2009

Time Allowed – 1 Hour 30 minutes
+ 5 minutes reading time

Remember to start each new question in a new booklet
Students must answer questions using a blue/black pen and/or a sharpened B or HB pencil.
Approved scientific calculators may be used
Students need to be aware that
- * 'bald' answers may not gain full marks.
- * untidy and/or poorly organised solutions may not gain full marks.

Marks

Question 1 Preliminary	/ 16	
Question 2 Integration	/ 16	
Question 3 Trigonometry	/ 16	
Question 4 Polynomials Circle Geometry	/ 10 / 5	
Total	/ 64	

Mathematics Extension 1 – Half Yearly Examination Year 12 – 2009

Question 1 (16 Marks) **Marks**

a) Solve $\dfrac{1}{2x-1} \le 2$ **3**

b) P divides the interval from (4,-1) to (0,3) in the ratio k:1 **3**
(i) Write down the coordinates of P.
(ii) If P lies on $x+2y=0$, find the coordinates of P.

c) Prove that, if $x^4 - x^3 + kx - 4$ has a factor of $(x+1)$, then it also has a factor of $(x-2)$. **3**

d) When the polynomial $P(x)$ is divided by x^2-1 the remainder is $3x-1$.
What is the remainder when $P(x)$ is divided by $x-1$? Justify your answer. **2**

e) (i) Show that the acute angle θ between the straight lines $y = x + 2$ and $y = mx+b$ is given by $\tan\theta = \left|\dfrac{m-1}{m+1}\right|$. **1**

(ii) Write down a similar result for the angle ϕ between the straight lines **1**

$y = 3x - 1$ and $y = mx+b$.

(iii) Hence find the gradient(s) m of the line(s) $y = mx+b$ bisecting the angles between the straight lines $y = x + 2$ and $y = 3x - 1$. **3**
(Answer in simplest surd form)

Question 2 (16 Marks) Start a New Booklet **Marks**

a) Evaluate $\int_{\frac{1}{2}}^{1} 4t(2t-1)^5 \, dt$ by using the substitution $u = 2t-1$ **3**

b) Use the substitution $u = x^2+2$ to evaluate $\int_0^1 \dfrac{x}{(x^2+2)^2} \, dx$ **3**

c) (i) Prove that $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x \, dx = \frac{\pi}{8} - \frac{1}{4}$. **2**

(ii) Find $\dfrac{d}{dx}(x\sin^2 x)$ **2**

(iii) Hence, or otherwise, prove $\displaystyle\int_0^{\frac{\pi}{4}} x\sin 2x \, dx = \frac{1}{4}$. **2**

d) (i) Divide x^3 *by* x^2+1 expressing your answer in the form $x^3 = (x^2+1)Q(x) + R(x)$. **2**

(ii) Hence, or otherwise, evaluate $\int_0^1 \dfrac{x^3}{x^2+1} \, dx$ **2**

Question 3 (16 Marks) (Start a New Booklet) **Marks**

a) Find in simplest exact form, the value of tan 105° **2**

b) Express $\cos x + \sqrt{3}\sin x$ in the form $A\cos(x-\alpha)$.
Hence solve $\cos x + \sqrt{3}\sin x = 1$ for $0 \le x \le 2\pi$. Give exact answers. **3**

c) (i) Prove that $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$. **1**

(ii) Hence evaluate $\int_0^{\frac{\pi}{3}} \sin 2x \cos x \; dx$ **3**

d) (i) Solve the equation $\sin 2x = \sin x$ for $0 \le x \le 2\pi$. **2**

(ii) Give general solutions to this equation. **2**

(iii) Show that if $0 < x < \frac{\pi}{3}$, then $\sin 2x > \sin x$. **1**

(iv) Find the area enclosed between the curves $y = \sin 2x$ and $y = \sin x$ **2**

for $0 \le x \le \frac{\pi}{3}$

Question 4 is on next page.

Question 4 (16 Marks) Start a New Booklet **Marks**

a) The function $f(x) = x^3 - \ln(x+1)$ has one root between 0.5 and 1.

(i) Show that the root lies between 0.8 and 0.9. **1**

(ii) Use halving the interval method to find the root correct to one decimal place. **1**

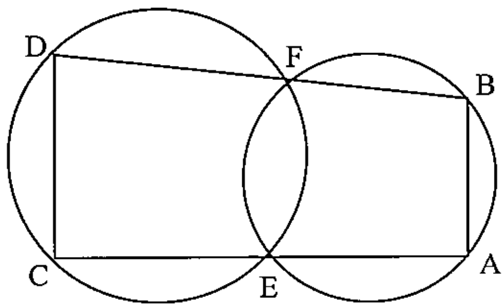
b) (i) Show that the function $f(x) = x^3 - x^2 - x - 1$ has a zero between 1 and 2. **1**

(ii) Taking $x = 2$ as a first approximation to this zero, use Newton's method to find a second approximation. **2**

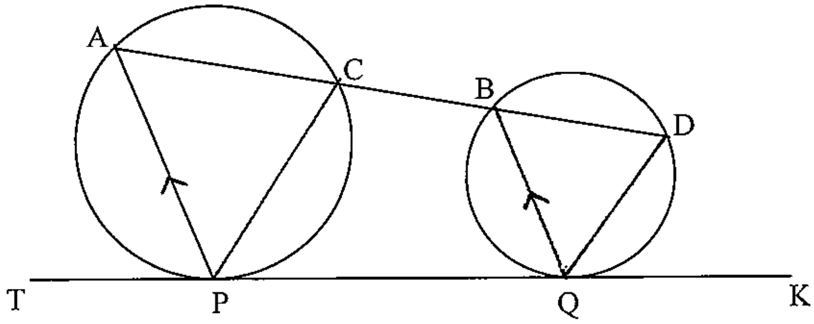
(iii) Give a geometrical interpretation of the process used in (ii). Why is $x = 1$ unsuitable as a first approximation to this zero? **2**

c) The polynomial $(x-\alpha)^3 + \beta$ is zero at $x = 1$, and, when divided by x, the remainder is -7. Find all possible values of the pair α *and* β. Hint: You may use the binomial expansion $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$. **3**

d) Two circles intersect at E and F. AEC and BFD are straight lines. Copy the diagram and prove that AB is parallel to CD. **2**



e) PQ is a common tangent to circles PAC and QBD
PA is parallel to QB.



(i) Prove that PC is parallel to QD. **2**

(ii) PQBC is a cyclic quadrilateral. **1**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

Solutions to 2009 Ext 1 H/Y Yr12

Question 1

a) If $x > \frac{1}{2}$ solve

$1 \le 4x - 2$

$3 \le 4x$

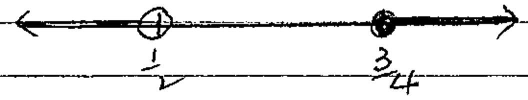
$\frac{3}{4} \le x$

If $x < \frac{1}{2}$ solve

$1 \ge 4x - 2$

$3 \ge 4x$

$\frac{3}{4} \ge x$



$x < \frac{1}{2}$ or $x \ge \frac{3}{4}$

b) i) $x = \frac{4}{k+1}$ $\quad y = \frac{3k-1}{k+1}$

ii) $\frac{4}{k+1} + \frac{6k-2}{k+1} = 0$

$6k + 2 = 0$

$k = -\frac{1}{3}$

P is $(6, -3)$

c) $P(-1) = 0 \quad \therefore 1 + 1 - k - 4 = 0$

$k = -2$

$P(x) = x^4 - x^3 - 2x - 4$

$P(2) = 16 - 8 - 4 - 4$

$= 0$

$\therefore x - 2$ is a factor

d) $P(x) = (x^2 - 1)Q(x) + 3x - 1$

$P(1) = 0 + 3 - 1$

$P(1) = 2$

remainder is 2.

e) For $y = x + 2$, $m_2 = 1$

For $y = mx + b$, $m_1 = m$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{m - 1}{1 + m}\right|$

ii) For $y = 3x - 1$, $m_2 = 3$

$\tan\phi = \left|\frac{m - 3}{1 + 3m}\right|$

iii) $\left|\frac{m-1}{m+1}\right| = \left|\frac{m-3}{3m+1}\right|$

$|m-1||3m+1| = |m-3||m+1|$

$|3m^2 - 2m - 1| = |m^2 - 2m - 3|$

$\therefore 3m^2 - 2m - 1 = m^2 - 2m - 3$

$2m^2 + 2 = 0$

No solutions

OR $3m^2 - 2m - 1 = -(m^2 - 2m - 3)$

$4m^2 - 4m - 4 = 0$

$m^2 - m - 1 = 0$

$m = \frac{1 \pm \sqrt{1+4}}{2}$

$m = \frac{1 \pm \sqrt{5}}{2}$

Question 2

a) $\int_{\frac{1}{2}}^{1} 4t(2t-1)^5\,dt$ $\quad u = 2t - 1$, $du = 2\,dt$, $t = \frac{1}{2}$ $u = 0$, $t = 1$ $u = 1$

$= \int_0^1 4.\frac{(u+1)}{2}\, u^5 \frac{du}{2}$

$= \int_0^1 u^6 + u^5\,du$

$= \left[\frac{u^7}{7} + \frac{u^6}{6}\right]_0^1 = \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$

b) $\int_0^1 \frac{x\,dx}{(x^2+2)^2}$ $\quad u = x^2 + 2$, $du = 2x\,dx$, $x = 0$ $u = 2$, $x = 1$ $u = 3$

$= \int_2^3 \frac{du}{2u^2}$

$= \left[\frac{1}{2}\,\frac{u^{-1}}{-1}\right]_2^3 = \left[\frac{-1}{2u}\right]_2^3$

$= -\frac{1}{6} + \frac{1}{4} = \frac{1}{12}$

c) i) $\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

2c(i) cont.

$\int_0^{\frac{\pi}{4}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} 1-\cos 2x\,dx$

$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{\pi}{8} - \frac{1}{4}\right) - (0-0)$ as required.

(ii) $\frac{d}{dx}(x\sin^2 x) = x.2\sin x\cos x + \sin^2 x$

$= x\sin 2x + \sin^2 x$

$\therefore \int x\sin 2x + \sin^2 x\,dx = x\sin^2 x$

(iii) $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx = \left[x\sin^2 x\right]_0^{\pi/4} - \int_0^{\pi/4}\sin^2 x\,dx$

$= \left(\frac{\pi}{4}\cdot\left(\frac{1}{\sqrt2}\right)^2 - 0\right) - \left(\frac{\pi}{8}-\frac{1}{4}\right)$

$= \frac{\pi}{8} - \frac{\pi}{8} + \frac{1}{4} = \frac{1}{4}$ as required

d)(i)

$$\begin{array}{r} x \\ x^2+1\ \overline{)\ x^3 + 0x^2 + 0x + 0} \\ -x^3 \qquad + x \\ \hline -x \end{array}$$

$\therefore x^3 = (x^2+1)\,x - x$

(ii) $\int_0^1 \frac{x^3}{x^2+1} = \int_0^1 x - \frac{x}{x^2+1}\,dx$

$= \left[\frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)\right]_0^1$

$= \left(\frac{1}{2} - \frac{1}{2}\ln 2\right) - \left(0 - \frac{1}{2}\ln 1\right)$

$= \frac{1}{2}(1-\ln 2)$

Question 3

a) $\tan 105^\circ = \tan(60^\circ+45^\circ)$

$= \frac{\tan 60^\circ + \tan 45^\circ}{1-\tan 60^\circ \tan 45^\circ}$

$= \frac{\sqrt3 + 1}{1-\sqrt3} \times \frac{1+\sqrt3}{1+\sqrt3}$

$= -\frac{1}{2}(1+2\sqrt3+3)$

$= -2-\sqrt3$

b) $\cos x + \sqrt3\sin x \equiv A\cos x\cos\alpha + A\sin x\sin\alpha$

$A\cos\alpha = 1$, $A\sin\alpha = \sqrt3$ $\Rightarrow \tan\alpha = \sqrt3$, $\alpha = \frac{\pi}{3}$

$A^2 = 1^2 + (\sqrt3)^2 \Rightarrow A = 2$

$2\cos\left(x-\frac{\pi}{3}\right) = 1$

$\cos\left(x-\frac{\pi}{3}\right) = \frac{1}{2}$

$x - \frac{\pi}{3} = -\frac{\pi}{3}$ or $\frac{\pi}{3}$ or $2\pi - \frac{\pi}{3}$

$x = 0$ or $\frac{2\pi}{3}$

c)(i) $\sin(\alpha+\beta) + \sin(\alpha-\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= 2\sin\alpha\cos\beta$ as required

(ii) Put $\alpha = 2x$ & $\beta = x$

$\int_0^{\frac{\pi}{3}} \sin 2x\cos x\,dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin 3x + \sin x\,dx$

$= \frac{1}{2}\left[-\frac{1}{3}\cos 3x - \cos x\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{2}\left\{\left(-\frac{1}{3}\cdot -1 - \frac{1}{2}\right) - \left(-\frac{1}{3} - 1\right)\right\}$

$= \frac{1}{2}\left\{-\frac{1}{6} + \frac{4}{3}\right\} = \frac{7}{12}$

d)(i) $2\sin x\cos x - \sin x = 0$

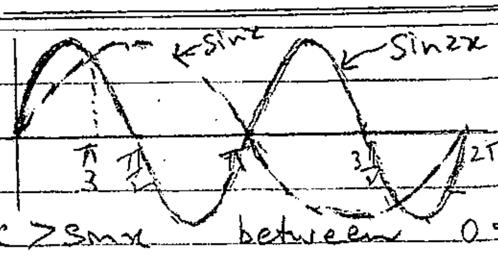
$\sin x(2\cos x - 1) = 0$

$\sin x = 0$ or $\cos x = \frac{1}{2}$

$x = 0, \pi, 2\pi$ $\quad x = \frac{\pi}{3}, \frac{5\pi}{3}$

(ii) $x = n\pi$ or $x = 2n\pi \pm \frac{\pi}{3}$

(iii)



$\sin 2x > \sin x$ between 0 & $\frac{\pi}{3}$

3(d) iv) Area $= \int_0^{\frac{\pi}{3}} \sin 2x - \sin x \, dx$

$= \left[-\frac{\cos 2x}{2} + \cos x\right]_0^{\frac{\pi}{3}}$

$= \left(-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0 + \cos 0\right)$

$= \frac{1}{4} + \frac{1}{2} + \frac{1}{2} - 1$

$= \frac{1}{4} u^2$

Question 4

a) $f(x) = x^3 - \ln(x+1)$

i) $f(0.8) = .8^3 - \ln 1.8$
$\doteqdot -0.0758$

$f(0.9) = .9^3 - \ln 1.9$
$\doteqdot 0.0871$

Function is defined & continuous in this region
∴ Root lies between 0.8 & 0.9

ii) $f(0.85) = .85^3 - \ln 1.85$
$\doteqdot -0.00106$

Root lies between 0.85 & 0.9. ∴ To one decimal place the root is 0.9.

b) i) $f(1) = 1 - 1 - 1 - 1 = -2$
$f(2) = 8 - 4 - 2 - 1 = 1$

Function is continuous & defined in the region.
One positive one negative. Root lies between 1 & 2.

ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f'(x) = 3x^2 - 2x - 1$
$f'(2) = 12 - 4 - 1 = 7$

$x_2 = 2 - \frac{1}{7}$

$x_2 = 1\frac{6}{7} \doteqdot 1.857$

iii) The tangent at $x = 2$ cuts x axis at $1\frac{5}{6}$. This is a better approximation provided there is no stationary point in the vicinity. $f'(1) = 0$
∴ Stationary at $x = 1$
$\left(x_2 = 1 - \frac{-2}{0} \text{ undefined}\right)$

c) $P(x) = (x - \alpha)^3 + \beta$
$P(1) = 0 = (1-\alpha)^3 + \beta$ — ①
$P(0) = -7 = (-\alpha)^3 + \beta$ — ②
$1 - 3\alpha + 3\alpha^2 - \alpha^3 + \beta = 0$
$1 - 3\alpha + 3\alpha^2 - 7 = 0$
$3\alpha^2 - 3\alpha - 6 = 0$
$(\alpha - 2)(\alpha + 1) = 0$
$\left.\begin{array}{l}\alpha = 2\\ \beta = 1\end{array}\right\}$ or $\left.\begin{array}{l}-1\\ -8\end{array}\right\}$

d) Join EF
∠DFE = ∠BAE (exterior angle of cyclic quad ABFE)
∠DCE = 180° − ∠DFE
(Opposite angles of cyclic quad are supplementary)
∴ ∠DCE + ∠BAE = 180°
But these are cointerior
∴ AB ∥ CD

e) ∠APT = ∠ACP } Angle between
∠BQP = ∠BDQ } tangent & chord equals angle in alternate segment
But ∠APT = ∠BQP
Corresp ∠'s AP ∥ BQ
∴ ∠ACP = ∠BDQ. But these are corresponding ∴ PC ∥ DQ

ii) ∠APT = ∠ACP (alt segment)
∠APT = ∠BQP (corresp ∠'s AP ∥ BQ)
∴ ∠ACP = ∠BQP. But this is exterior ∠ ∴ PQBC is cyclic.