Student Name: ....

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2010

YEAR 12 ASSESSMENT TASK 2

Mathematics Extension 1

General Instructions

- Working Time – 1 1/2 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a new sheet of paper.

Total Marks - 60

Attempt Questions 1-4

All questions are of equal value.

Question 1 (15 Marks) **Marks**

(a) Divide the interval from A (-1, 5) to B (6, -4) *externally* in the ratio 3:2 **2**

(b) Simplify $\dfrac{4^n}{4^{n+1}-4^n}$ **2**

(c) Solve the inequality $\dfrac{2x-3}{x-2} \geq 1$ and graph your solution on the number line **3**

(d) Find $\int x\sqrt{1-x}\, dx$ using the substitution $u = 1 - x$ **3**

(e) Use the substitution $t = u^2 - 1$ to evaluate $\int_0^3 \dfrac{t}{\sqrt{1+t}}\, dt$ **3**

(f) The curves $y = x^3$ and $y = \dfrac{x^2+3}{4}$ meet at the point $(1,1)$

Find the angle between the tangents to the curves at this point **2**

End of Question 1

Question 3 (15 Marks) Begin a new page **Marks**

(a) Find the value of k if $\int_0^k \sqrt{1+x}\,dx = \frac{14}{3}$ **3**

(b) The function $f(x) = x^3 - 5x^2 - 24x + 118$

Taking $x = 5$ as the first approximation of the root of the equation

(i) Use one application of Newton's Method to obtain a second approximation to a root of the equation. **2**

(ii) Explain this result by drawing a sketch of $y = f(x)$ near $x = 5$ stating the sign of $P(5), P'(5), P''(5)$ **3**

(c) Find the simultaneous solution of $|x-3| < 4$ and $|x-1| > 1$ **3**

(d) Let $P(x) = -2x^3 + px^2 - qx + 5$

(i) Show that if $P(x)$ is to have any stationary points then $p^2 - 6q \geq 0$ **2**

(ii) Explain the situation for $p^2 - 6q = 0$ **2**

End of Question 3

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

SOLUTIONS.

QUESTION 1

a) $(-1,5)(6,-4)$

$3:-2$

$x = \frac{2+18}{1}$ $\quad y = \frac{-10-12}{1}$

$= 20 \quad y = -22.$

$\therefore P(20,-22)$

(b) $\frac{4^n}{4^n(4-1)} = \frac{1}{3}$

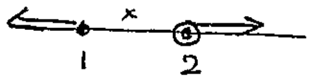
(c) $\frac{2x-3}{x-2} \geq 1$

critical point $x=2$.

$\frac{2x-3}{x-2} = 1$

$2x-3 = x-2$

$x = 1$



test $x=0$

$\frac{-3}{-2} \geq 1$ true

$x \leq 1, \; x > 2$

d) $u = 1-x$

$\frac{du}{dx} = -1$

$du = -dx.$

$\int (1-u) \cdot \sqrt{u} \,.\, -du$

$-\int u^{1/2} - u^{3/2} \, du$

$-\left[\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right] + c$

$-\frac{2}{3}(1-x)^{3/2} + \frac{2}{5}(1-x)^{5/2} + c$

e) $t = u^2 - 1$

$\frac{dt}{du} = 2u$

$dt = 2u.du$

$t=1 \quad u=\sqrt{2}$

$t=0 \quad u=1$

$\int_1^{\sqrt{2}} \frac{u^2-1}{\sqrt{u^2}} . \, 2u \, du$

$\int_1^{\sqrt{2}} (u^2-1) \times 2 \, du$

$2\int_1^{\sqrt{2}} (u^2-1) \, du$

$= 2\left[\frac{u^3}{3} - u\right]_1^{\sqrt{2}}$

$2\left[\left(\frac{2\sqrt{2}}{3} - \sqrt{2}\right) - \left(\frac{1}{3} - 1\right)\right]$

$2\left[\frac{2}{3} - \frac{\sqrt{2}}{3}\right]$

$\frac{4-2\sqrt{2}}{3}$

f) $y = x^3$

$\frac{dy}{dx} = 3x^2$

at $x=1$

$m_1 = 3$

$y = \frac{x^2}{4} + \frac{3}{4}$

$\frac{dy}{dx} = \frac{2x}{4}$

at $x=1$

$m_2 = \frac{1}{2}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{3 - \frac{1}{2}}{1 + 3 \times \frac{1}{2}}\right|$

$= 1$

$\therefore \theta = 45^\circ$

QUESTION 2.

$f(2) = (2)^3 - 4(2^2) + 7(2) - 6$

$= 8 - 16 + 14 - 6$

$= 0$

$\therefore x-2$ is a factor

ii) $\dfrac{x^3-4x^2+7x-6}{x-2} = x^2-2x+3$

this is a quadratic with $\Delta = 4-12$
$= -8$

$\Delta < 0$ no real roots

$\therefore x = 2$ is the only real root.

b). $f(x) = x^3+5x^2+17x-10$

$f(0) = -10 < 0$

$f(2) = 8+20+34-10$

$= 52 > 0$

curve is continuous and there is a sign change so root lies between $x=0$ and 2

ii) use $x=1$

$P(1) = 1+5+17-10$

$= 13 > 0$

root lies between $x=0$ and $x=1$

iii) $P(\frac{1}{2}) = \frac{1}{8} + \frac{10}{8} + \frac{68}{8} - 10$

$= \frac{79}{8} - 10 < 0$

The root lies between $x=\frac{1}{2}$ and $x=1$

closer to 1 than to 0

c)

i) $P(3) = 27 - 9(k+1) + 3k + 12$

$= 27 - 9k - 9 + 3k + 12$

$= 30 - 6k.$

ii) If its divisible then $P(3) = 0$

$30 - 6k = 0$

$k = 5.$

iii) $P(x) = x^3 - 6x^2 + 5x + 12.$

$= (x-3)(x^2-3x-4)$

$= (x-3)(x-4)(x+1)$

roots $x=3,\ x=4,\ x=-1$

d) $5^x - 24 = 5^{2-x}$

$5^x - 24 = \dfrac{5^2}{5^x}$

$5^{2x} - 24.5^x - 25 = 0$

let $m = 5^x$

$m^2 - 24m - 25 = 0$

$(m-25)(m+1) = 0$

$m = 25 \quad m = -1$

$5^x = 25 \qquad 5^x = -1$

$x = 2 \qquad$ no solution.

QUESTION 3.

a) $\left[\dfrac{(1+x)^{3/2}}{3/2}\right]_0^k = \dfrac{14}{3}$

$\left[\dfrac{2}{3}(1+x)^{3/2}\right]_0^k = \dfrac{14}{3}$

$\left[(1+x)^{3/2}\right]_0^k = 7$

$(1+k)^{3/2} - 1 = 7$

$(1+k)^{3/2} = 8$

$1+k = 4$

$k = 3$

(b) $f(x) = x^3 - 5x^2 - 24x + 118$ $\quad f(5) = -2$

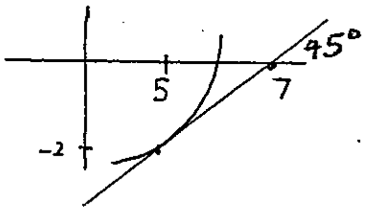
$f'(x) = 3x^2 - 10x - 24$ $\quad f'(5) = 1$

$x_2 = x - \dfrac{f(x)}{f'(x)}$

$= 5 - \dfrac{-2}{1}$

$= 7$

ii) $f(5)$ $\quad f'(5)$ $\quad f''(5)$



the tangent at $x=5$ cuts the x axis further from the root $x=r$ than first approx $x=5$, due to curvature & concavity $\therefore$ $x=5$ is better ...

c) $|x-3| < 4$ $\qquad |x-1| > 1$

$-4 < x-3 < 4$ $\qquad x-1 > 1$ or $x-1 < -1$

$-1 < x < 7$ $\qquad x > 2$ or $x < 0$

$-1 \quad 0 \quad 2 \quad 7$

$2 < x < 7$ or $-1 < x < 0$

d) i) $P(x) = -2x^3 + px^2 - qx + 5$

$P'(x) = -6x^2 + 2px - q$

for stat pt $P'(x) = 0$. with this quadratic

$\Delta \geq 0$ to have real solutions

$\therefore$ for a point to exist $4p^2 - 24q \geq 0$

(ii) If $\Delta = 0$

$x = \dfrac{-2p}{-12}$

double roo[t]

so horizon[tal]

inflexion p[t]

Question 4

a) $1\times 2 + 3\times 4 + \ldots (2n-1)\times 2n = \frac{1}{3}n(n+1)(4n-1)$

step 1 prove true for $n=1$

$LHS = 1\times 2 = 2$ $\quad RHS = \frac{1}{3}\cdot 1\cdot 2\cdot 3 = 2$ $\quad \therefore$ true for $n=1$

assume true for $n=k$

$1\times 2 + 3\times 4 + \ldots (2k-1)\times 2k = \frac{1}{3}k(k+1)(4k-1)$

prove true for $n=k+1$

$$\underbrace{1\times 2 + 3\times 4 + 5\times 6 + \ldots (2k-1)\times 2k} + (2k+1)\cdot 2\cdot(k+1)$$

$$= \frac{1}{3}k(k+1)(4k-1) + (2k+1)\times 2(k+1)$$

$$= \frac{1}{3}\underline{k(k+1)(4k-1) + 6(2k+1)(k+1)}$$

$$= \frac{1}{3}\underline{(k+1)[k(4k-1) + 6(2k+1)]}$$

$$= \frac{1}{3}\underline{(k+1)[4k^2 + k + 6]}$$

$$= \frac{1}{3}(k+1)[k+2][4k+3]$$

$$= \frac{1}{3}(k+1)(k+2)(4(k+1)-1)$$

since it is true for $n=1$ it is true for $n = 1+1 = 2$. Since it is true for $n=k$ it is true for $n=k+1$ and hence all positive integer values for n.

b) i) $PS = \sqrt{(2t-0)^2 + (t^2-1)^2}$

$$= \sqrt{t^4 + 2t^2 + 1}$$

$$= t^2 + 1$$

$PM = t^2 + 1$ (definition of parabola)

$\therefore \Delta PSM$ is isosceles.

ii) $y = \frac{x^2}{4}$

$\frac{dy}{dx} = \frac{2x}{4}$

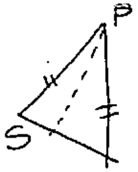
at $P(x = 2t$

$m = t$.

$S(0,1) \quad M(2t,-1)$

$m_{SM} = \frac{1-{-1}}{0-2t} = \frac{2}{-2t} = \frac{-1}{t}$

the two lines are perpendicular

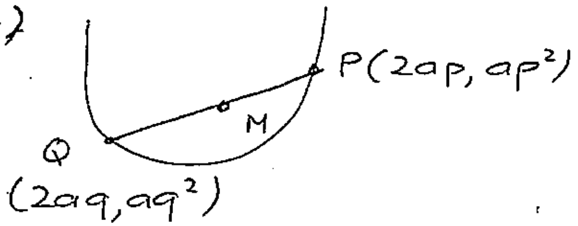
iii) 

the altitude of isosc Δ bisects vertex

$\therefore \angle SPC = \angle CPm.$

also $\angle CPm$ & $\angle APB$ are vert opp & equal

$\therefore \angle APB = \angle SPC.$

c) 

i) $M = \dfrac{2ap+2aq}{2}\ ,\ \dfrac{ap^2+aq^2}{2}$

$= a(p+q) \qquad = \dfrac{a}{2}(p^2+q^2)$

ii) equation of PQ

$y - ap^2 = \dfrac{ap^2-aq^2}{2ap-2aq}(x-2ap)$

since through $(0, 4a)$

$4a - ap^2 = \dfrac{ap^2-aq^2}{2a(p-q)}(0-2ap)$

$4a - ap^2 = \dfrac{p+q}{2}(-2ap)$

$8a - 2ap^2 = -2ap^2 - 2apq$

$8\cancel{a} = -2\cancel{a}pq$

iii) for M $\quad x = a(p+q) \quad y = \dfrac{a}{2}(p^2+q^2)$

so $\dfrac{x}{a} = (p+q) \qquad y = \dfrac{a}{2}\left[(p+q)^2 - 2pq\right]$

$y = \dfrac{a}{2}\left[\left(\dfrac{x}{a}\right)^2 - 2\times -4\right]$

$y = \dfrac{a}{2}\left[\dfrac{x^2}{a^2} + 8\right]$ *

<u>OR</u> $\quad 2y = \dfrac{x^2}{a} + 8a$ *

$2ay = x^2 + 8a^2$ *