GOSFORD HIGH SCHOOL

Year 12

2011

HSC

MATHEMATICS EXTENSION I

Assessment Task #2

Time Allowed: 90 minutes+5 minutes reading time

Instructions:

- Start each question on a new sheet of paper.
- Attempt questions 1-4.
- Board approved calculators may be used.
- Write using black or blue pen.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question.

QUESTION 1: (16 Marks)

Marks

a. A(1,0) and B(2,4) are two points on the number plane. Find the co-ordinates of the point P that divides the interval AB externally in the ratio 3:1 — 2

b. Sketch the region on the number plane that satisfies $y \leq \sqrt{4 - x^2}$ — 2

c. Find the number of ten letter arrangements that can be made from the letters of the word 'MOOLOOLABA'. — 1

d. i. Find the exact values of the gradients of the tangents to the curve $y = e^x$ at the points where $x = 0$ and $x = 1$. — 1

ii. Hence, find the acute angle between these tangents, correct to the nearest degree. — 2

e. Solve: — 2

$$\frac{5}{|2x - 1|} \leq 1$$

f. Evaluate $\lim_{x \to 0} \frac{3x}{\sin 4x}$ — 1

Question 1 Continued: **Marks**

g. A meeting room contains a round table surrounded by twelve chairs. If a committee of twelve includes five women, how many arrangements around the table are there in which all the women sit together? 2

h. Use one iteration of Newton's method to find an approximation, to the root of the equation: 3

$$x \ln x - 2x = 0, \qquad near\ x = 7$$

(Answer correct to 3 significant figures.)

End of Question 1

QUESTION 2: (18 Marks) Use a new sheet of paper

Marks

a. i. Write down the expansion of sin(A – B) 1

ii. Hence, find the exact value of $sin\frac{\pi}{12}$ 2

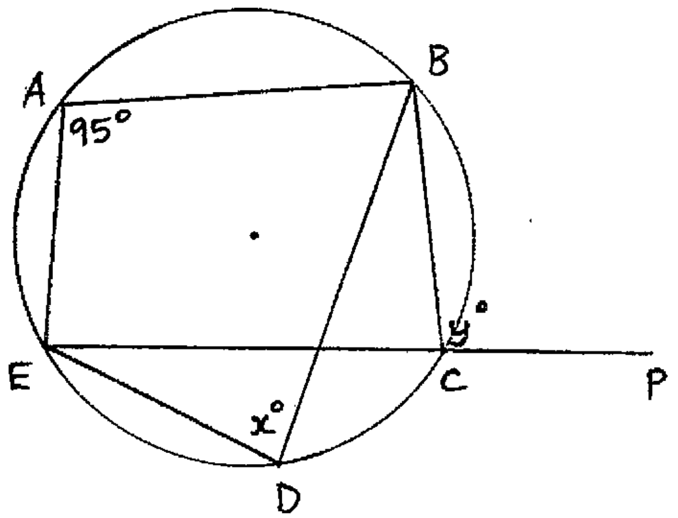
(Express your answer with a rational denominator.)

b. Find **ALL** values of θ *for which* $sec\ \theta = \frac{2}{\sqrt{3}}$ 1

c. By factorising first, give the general solution, in terms of π , to the equation 2

$$\sqrt{3}\ tan^2\theta - \sqrt{3}tan\theta - tan\theta + 1 = 0$$

d. Find the value of the pronumerals, giving reasons for your answers: 2



Question 2 continued: **Marks**

e. Let $P(x) = (x+1)(x-3)Q(x) + a(x+1) + b$,
where Q(x) is a polynomial and a and b are real numbers.

When P(x) is divided by (x+1) the remainder is 11 and when P(x) is divided by (x – 3) the remainder is 1.

i. What is the value of b? 1

ii. What is the remainder when P(x) is divided by (x+1) (x – 3) 2

f. If $\propto, \beta$ *and* γ are the roots of the equation: 2

$$x^3 - x^2 + 4x - 1 = 0$$

Evaluate $(\propto +1)(\beta + 1)(\gamma + 1)$

g. If the roots of the equation $x^3 - 2x^2 - x - p = 0$ are $\propto, \propto +1$ *and* $\propto +3$ find the value of p 2

h. Without the use of calculus, sketch the curve $y = \frac{x-1}{x^2-9}$ 3

Clearly indicate on your sketch all asymptotes and intercepts.

End of Question 2

QUESTION 3: (18 Marks) Use a new sheet of paper

Marks

a. Prove, by mathematical induction, that $n^3 + 2n$ is divisible by 12 for all even positive integers. 3

b. i. Show that $f(x) = x^4 - 10$ has a root between $x = 1$ and $x = 2$. 1

ii. Hence, use the method of halving the interval to show that $\sqrt[4]{10}$ lies between 1.75 and 1.875 2

c. Find $\int 2sin^2 3x\, dx$ 2

d. Use the substitution $x = u^2 - 1$ to evaluate 3

$$\int_0^3 \frac{x}{\sqrt{x+1}}\, dx$$

e. Use the table of standard integrals on page 10 to show that 2

$\int_0^a \frac{1}{\sqrt{x^2+a^2}}\, dx$ is independent of a

Question 3 continued: **Marks**

f. The region bounded by the curve

$$y = 1 + cosx, \text{ the } x \text{ axis, } x = 0 \text{ and } x = \pi$$

is rotated about the x axis to form a solid.

i. Show that the volume of the solid obtained is given by: 1

$$V = \pi \int_0^{\pi} 1 + 2cosx + cos^2 x \, dx$$

ii. Hence, find the exact volume of the solid. 2

g. Use the substitution $u = x - 1$ to find 2

$$\int 3x(1-x)^3 dx$$

End of Question 3

QUESTION 4: (18 Marks) Use a new sheet of paper

Marks

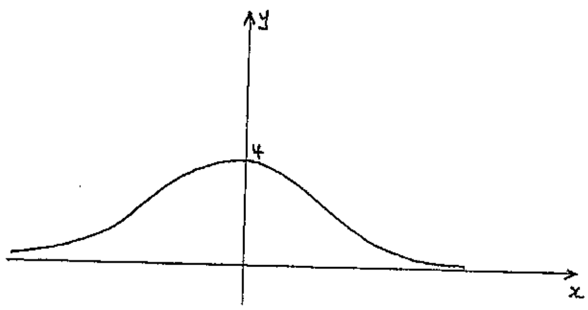
a. Consider the function $f(x) = \log_e (x - 4)$

i. Find the inverse function $f^{-1}(x)$ 1

ii. State the range of $f^{-1}(x)$ 1

iii. On the same set of axes sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, clearly indicating all asymptotes and intercepts. ~~1~~ 2

b. Given the sketch of $g(x) = \frac{4}{1+x^2}$



i. Copy this sketch.

ii. State the largest domain containing $x = -1$ for which $g(x)$ has an inverse function. 1

iii. Let $g^{-1}(x)$ be the inverse function corresponding to this restricted domain. What is the domain of $g^{-1}(x)$ 1

iv. On the same set of axes in (i) sketch $y = g^{-1}(x)$ 1

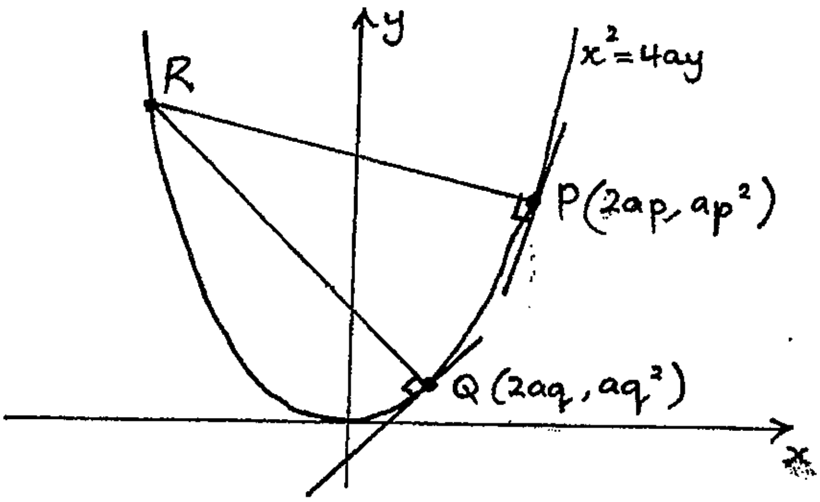
v. Hence, find the equation of the inverse function $g^{-1}(x)$ 2

Question 4 continued: **Marks**

c. The function $h(x) = \frac{x+4}{x+1}$ is defined for $x > -1$
Let $h^{-1}(x)$ be the inverse function corresponding to the domain of $h(x)$.

Find all values of x for which $h(x) = h^{-1}(x)$ 2

d. P and Q are points on the parabola $x^2 = 4ay$ as shown in the diagram:



Normals from P and Q intersect at R. You may use the equation of the normal from P to be: $x + py = ap^3 + 2ap$ (DO NOT PROVE THIS!)

i. Show that the normals from P and Q intersect at the point 3
$(-apq(p+q), a(p^2 + pq + q^2 + 2)$

ii. If this point of intersection R also lies on the parabola show that $pq = 2$ 2

iii. Hence, find the cartesian equation of the locus of the midpoint M of PQ 2

END OF TEST

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

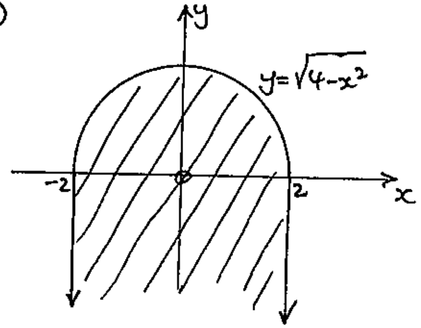
YEAR 12 2011 Assess #2 Maths Ext1 Solutions

Q1 (a) $P = \left(\frac{1\times -1 + 3\times 2}{3-1}, \frac{0\times -1 + 3\times 4}{3-1}\right)$

$= \left(\frac{-1+6}{2}, \frac{12}{2}\right)$

$= \left(\frac{5}{2}, 6\right)$

(b)



(c) Arrangements $= \frac{10!}{4!\,2!\,2!}$

$= 37800$

(d) (i) $\frac{dy}{dx} = e^x$

(when $x=0$) $\frac{dy}{dx} = 1 \rightarrow m_1$

(when $x=1$) $\frac{dy}{dx} = e \rightarrow m_2$

$\therefore$ gradients of tangents are 1 (at $x=0$) and e (at $x=1$)

(ii) using $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{1-e}{1+e}\right|$

$\therefore \tan\theta = 0.4621..$ (calc)

$\theta = 24.801...$ (calc)

$\therefore$ acute angle $= 25°$

(e) $\frac{5}{|2x-1|} \le 1$ CV: $x \ne \frac{1}{2}$

Consider $\frac{5}{|2x-1|} = 1$

$\therefore 5 = |2x-1|$

$2x-1 = 5$ or $2x-1 = -5$

$2x = 6$ $2x = -4$

$x = 3$ $x = -2$

Test
✓ × × ✓
-2 $\frac{1}{2}$ 3

$\therefore$ Solns: $x \le -2,\ x \ge 3$

(f) $\lim_{x\to 0} \frac{3x}{\sin 4x} = \frac{3}{4}$

(g) Consider the 5 women as one group $\therefore$ 8 around a circular table $= 7!$

Ways in which women arranged $= 5!$

$\therefore$ Total ways $= 7! \times 5!$

$= 604800$

(h) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $\left\{ \begin{array}{l} f(x) = x\ln x - 2x \\ f'(x) \\ = \ln x + 1 - 2 \\ = \ln x - 1 \end{array} \right.$

$x_2 = 7 - \frac{7\ln 7 - 14}{\ln 7 - 1}$

$= 7.40$

Q2 (a) (i)

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

(ii) Since $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$\sin\frac{\pi}{12} = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$

$= \frac{1}{\sqrt2}\times\frac{\sqrt3}{2} - \frac{1}{\sqrt2}\times\frac{1}{2}$

$= \frac{\sqrt3 - 1}{2\sqrt2}$

$= \frac{\sqrt3-1}{2\sqrt2}\times\frac{\sqrt2}{\sqrt2}$

$= \frac{\sqrt6 - \sqrt2}{4}$

(b) $\cos\theta = \sqrt3/2$

$\therefore \theta = 2\pi n \pm \frac{\pi}{6}$

(c) $\sqrt3\tan^2\theta - \sqrt3\tan\theta - \tan\theta + 1 = 0$

$\sqrt3\tan\theta(\tan\theta - 1) - (\tan\theta - 1) = 0$

$(\sqrt3\tan\theta - 1)(\tan\theta - 1) = 0$

$\therefore \tan\theta = \frac{1}{\sqrt3}$ or $\tan\theta = 1$

$\therefore \theta = n\pi + \frac{\pi}{6}$ or $\theta = n\pi + \frac{\pi}{4}$

(d) $x = 85$ (opposite $\angle$'s in a cyclic quadrilateral are supplementary)

Similarly $B\hat{C}E = 85°$

$\therefore y = 95$ (adjacent supplementary $\angle$'s)

(e) (i) $P(-1) = 11$ $\therefore 11 = 0 + 0 + b$

$\therefore b = 11$

(ii) $P(3) = 1$

$\therefore 1 = 0 + 4a + 11$

$-10 = 4a$

$-\frac{5}{2} = a$

$\therefore$ when $P(x)$ is divided by $(x+1)(x-3)$

the remainder is $a(x+1) + b$

i.e. $-\frac{5}{2}(x+1) + 11$

$= -\frac{5}{2}x - \frac{5}{2} + 11$

$= -\frac{5x}{2} + \frac{17}{2}$

(f) $\alpha + \beta + \gamma = 1$

$\alpha\beta + \alpha\gamma + \beta\gamma = 4$

$\alpha\beta\gamma = 1$

$(\alpha+1)(\beta+1)(\gamma+1) = (\alpha\beta + \alpha + \beta + 1)(\gamma + 1)$

$= \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \alpha + \beta\gamma + \beta + \gamma + 1$

$= \alpha\beta\gamma + (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) + 1$

$= 1 + 4 + 1 + 1$

$= 7.$

Q2 (g)

$\alpha + (\alpha+1) + (\alpha+3) = -\frac{b}{a}$

$3\alpha + 4 = 2$

$3\alpha = -2$

$\alpha = -\frac{2}{3}$

$\therefore$ if $P(x) = x^3 - 2x^2 - x - p$

$P(-\frac{2}{3}) = (-\frac{2}{3})^3 - 2(-\frac{2}{3})^2 - (-\frac{2}{3}) - p$

$= -\frac{8}{27} - \frac{8}{9} + \frac{2}{3} - p$

and since $P(-\frac{2}{3}) = 0$.

$-\frac{14}{27} - p = 0$

$\therefore p = -\frac{14}{27}$

(h) $y = \frac{x-1}{x^2-9}$

Vertical Asymptotes: $x = \pm 3$

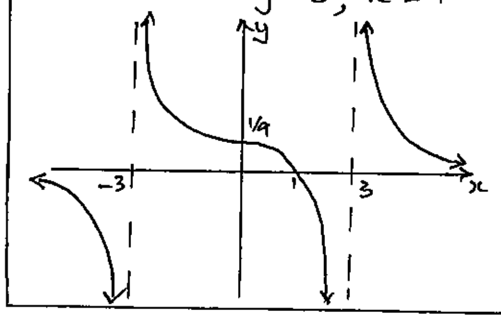
Horizontal Asymptotes:

as $x \to \infty$ $y \to 0^+$

$x \to -\infty$ $y \to 0^-$

Intercepts: $x = 0$, $y = \frac{1}{9}$

$y = 0$, $x = 1$



Q3 Step 1: to prove true for $n=2$

i.e $2^3 + 2(2) = 12$ which is divisible by 12

$\therefore$ true for $n = 2$

Step 2 Assume true for $n = 2k$ (where k is a positive integer)

i.e $\frac{(2k)^3 + (2k)2}{12} = M$ (where M is a positive integer)

i.e $8k^3 + 4k = 12M$

Step 3 to prove true for $n = 2k+2$

i.e $(2k+2)^3 + 2(2k+2)$ is divisible by 12.

$(2k+2)^3 + 2(2k+2)$

$= 8(k+1)^3 + 4(k+1)$

$= 8(k^3+3k^2+3k+1) + 4k + 4$

$= 8k^3 + 24k^2 + 24k + 8 + 4k + 4$

$= 8k^3 + 24k^2 + 28k + 12$

$= 8k^3 + 4k + 24k^2 + 24k + 12$

$= 12M + 24k^2 + 24k + 12$

$= 12(M + 2k^2 + 2k + 1)$

which is divisible by 12

$\therefore$ if true for $n = 2k$ then true for $n = 2k+2$

$\therefore$ true by induction for all even positive integers.

(b) (i) $f(1) = -9 < 0$

$f(2) = 6 > 0$

$\therefore$ there is a root between $x = 1$ and $x = 2$

(ii) if $x^4 - 10 = 0$

$x^4 = 10$

$\therefore x = \sqrt[4]{10}$

is a root of $f(x) = 0$

Since a root lies between $x = 1$ and $x = 2$

Let $x_1 = 1.5$

$f(1.5) < 0$

$\therefore$ Root lies between $x = 1.5$ and $x = 2$

Let $x_2 = 1.75$

$f(1.75) < 0$ $\therefore$ root lies between $x = 1.75$ and $x = 2$

Let $x_3 = 1.875$

$f(1.875) > 0$

$\therefore$ Root lies between $x = 1.75$ and $x = 1.875$ as required.

(c) $\int 2\sin^2 3x \, dx$

$= 2\int \frac{1}{2}(1 - \cos(6x)) \, dx$

$= \int 1 - \cos 6x \, dx$

$= x - \frac{1}{6}\sin 6x + c$

(d) $\int_0^3 \frac{x}{\sqrt{x+1}} \, dx$ $x = u^2 - 1$, $\frac{dx}{du} = 2u$, $dx = 2u\,du$

$= \int_1^2 \frac{u^2-1}{\sqrt{u^2-1+1}} . 2u\,du$ $x=3$ $u=2$, $x=0$ $u=1$

$= \int_1^2 \frac{2u^3 - 2u}{\sqrt{u^2}} \, du$

$= \int_1^2 2u^2 - 2 \, du$

$= \left[\frac{2u^3}{3} - 2u\right]_1^2$

$= (\frac{16}{3} - 4) - (\frac{2}{3} - 2)$

$= 1\frac{1}{3} - (-1\frac{1}{3})$

$= 2\frac{2}{3}$

(e) $\int_0^a \frac{1}{\sqrt{x^2+a^2}} \, dx$

$= \left[\ln\left|x + \sqrt{x^2+a^2}\right|\right]_0^a$

$= \ln\left|a + \sqrt{2a^2}\right| - \ln\left|\sqrt{a^2}\right|$

$= \ln\left|\frac{a + \sqrt{2}a}{a}\right|$

$= \ln\left|1+\sqrt{2}\right| = \ln(1+\sqrt{2})$

which is independent of a

(f) (i) $V = \pi\int_0^{\pi} y^2\,dx$

$= \pi\int_0^{\pi}(1+\cos x)^2\,dx$

$= \pi\int_0^{\pi} 1+2\cos x+\cos^2 x\,dx$

as required

(ii) $V = \pi\int_0^{\pi} 1+2\cos x+\frac{1}{2}(1+\cos 2x)\,dx$

$= \pi\left[x+2\sin x+\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{\pi}$

$= \pi\left[\left(\pi+0+\frac{\pi}{2}+0\right)-0\right]$

$= \frac{3\pi^2}{2}$ units^3

(g) $\int 3x(1-x)^3\,dx$

$u = x-1 \Rightarrow x = u+1$

$\frac{du}{dx} = 1$

$du = dx$

$\int 3(u+1)(-u)^3\,du$

$= -3\int u^4+u^3\,du$

$= -3\left(\frac{u^5}{5}+\frac{u^4}{4}\right)+c$

$= -3\left(\frac{(x-1)^5}{5}+\frac{(x-1)^4}{4}\right)+c$

Q4 (a)(i) $y = \log_e(x-4)$

$x = \log_e(y-4)$

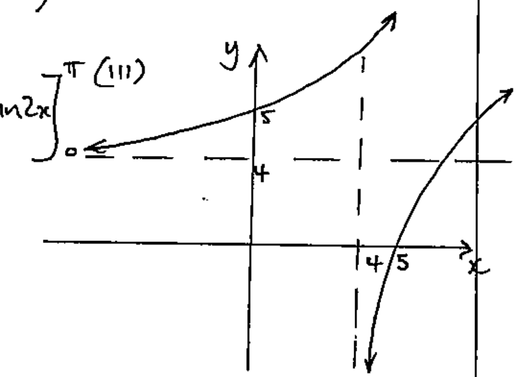
$e^x = y-4$

$e^x+4 = y$

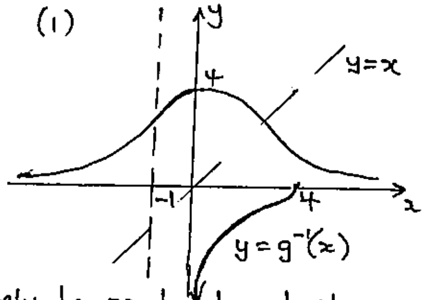
(ii) Range of $f^{-1}(x)$

$: y > 4$

(iii)



(b) (i)



(ii) Apply horizontal line test

largest restricted domain is $x \le 0$

(iii) Domain of $g^{-1}(x)$ corresponds to range of $g(x)$

i.e. $0 < x \le 4$

(iv) on above sketch

(v) $y = \frac{4}{1+x^2}$

$x = \frac{4}{1+y^2}$

$1+y^2 = \frac{4}{x}$

$y^2 = \frac{4}{x}-1$

$y^2 = \frac{4-x}{x}$

$y = \pm\sqrt{\frac{4-x}{x}}$

Since y is negative.

$y = -\sqrt{\frac{4-x}{x}}$

(c) Point of intersection $h(x)$ and $h^{-1}(x)$

Solve $y = h(x)$ and $y = x$

$\therefore \frac{x+4}{x+1} = x$

$x+4 = x^2+x$

$0 = x^2-4$

$\pm 2 = x$

(Since $x > -1$) $x = 2$

is the only solution.

(d) Normal at P:

$x+py = ap^3+2ap \quad -①$

Normal at Q

$x+qy = aq^3+2aq \quad -②$

① − ② : $(p-q)y = a(p^3-q^3)+2a(p-q)$

$y = a(p^2+pq+q^2)+2a$

$= a(p^2+pq+q^2+2)$

Sub in ① :

$x + ap(p^2+pq+q^2+2) = ap^3+2ap$

$x = ap^3+2ap-ap^3-ap^2q-apq^2-2ap$

$= -apq(p+q)$

$\therefore$ Pt of intersection: $\left(-apq(p+q),\ a(p^2+pq+q^2+2)\right)$

as required.

(ii) If R lies on $x^2 = 4ay$

$\therefore \left[-apq(p+q)\right]^2 = 4a\left[a(p^2+pq+q^2+2)\right]$

$a^2p^2q^2(p+q)^2 = 4a^2(p^2+pq+q^2+2)$

$p^2q^2(p+q)^2 = 4(p^2+pq+q^2)+8$

$p^2q^2(p+q)^2 = 4(p^2+2pq+q^2)+8-4pq$

$p^2q^2(p+q)^2 = 4(p+q)^2+4(2-pq)$

$p^2q^2(p+q)^2-4(p+q)^2-4(2-pq) = 0$

$(p+q)^2\left[p^2q^2-4\right]-4(2-pq) = 0$

$(p+q)^2(pq-2)(pq+2)-4(2-pq) = 0$

$(pq-2)\left[(p+q)^2(pq+2)+4\right] = 0$

$\therefore pq = 2$. (or show LHS = RHS)

(iii) $M = \left(a(p+q),\ \frac{ap^2+aq^2}{2}\right)$

$\therefore x = a(p+q) \quad -①$

$y = \frac{a}{2}(p^2+q^2) \quad -②$

From ① $\frac{x}{a} = p+q$ ② ⇒ $y = \frac{a}{2}\left[(p+q)^2-2pq\right]$

② ⇒ $y = \frac{a}{2}\left[\left(\frac{x}{a}\right)^2-4\right]$ $\therefore \frac{2y}{a} = \frac{x^2}{a^2}-4$

$2ay = x^2-4a^2$ $\therefore x^2 = 2a(y+2a)$