Name______________________ Teacher____________________

GOSFORD HIGH SCHOOL

2012

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 2

MATHEMATICS – EXTENSION 1

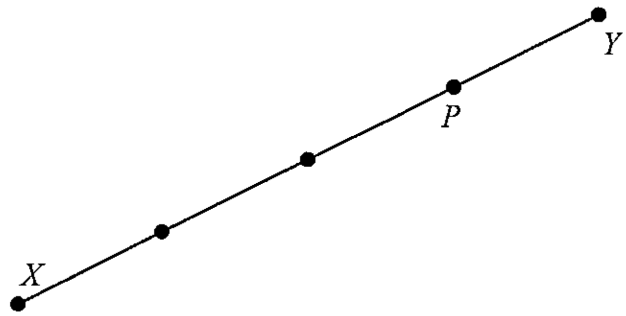
Duration- 90 minutes plus 5 minutes reading time

Question 1 (Multiple choice) **6** questions worth **1 mark each.** (Answer this section on the **test paper. Circle the correct answer)**	/6
Questions 2 to 7 **(free response questions)** These must be answered on your **own paper**. Start a new page for **each** question. Each question must be stapled separately Each question worth **9 marks.**	
Question 2 Preliminary topics	/9
Question 3 Iterative methods/polynomials	/9
Question 4 Mathematical induction	/9
Question 5 Parametric representation	/9
Question 6 Methods of integration	/9
Question 7 Circle Geometry	/9
TOTAL	**/60**

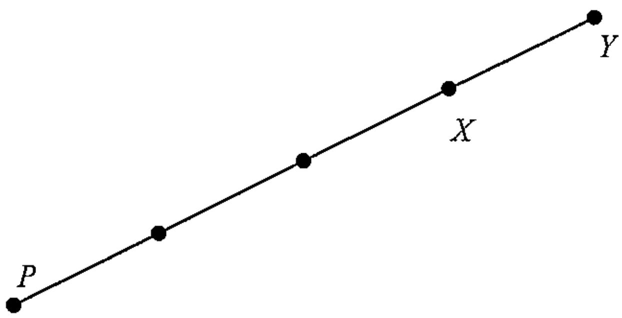
QUESTION 1 **6 marks** ***(start a new page)***

i) Which one of the following diagrams best represents a point P dividing an interval XY externally in the ratio 3 : 1

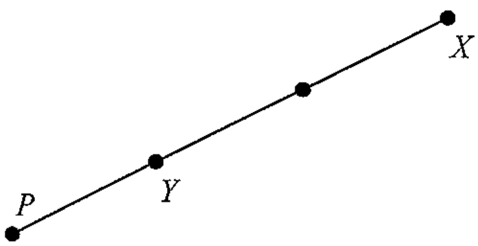
A)



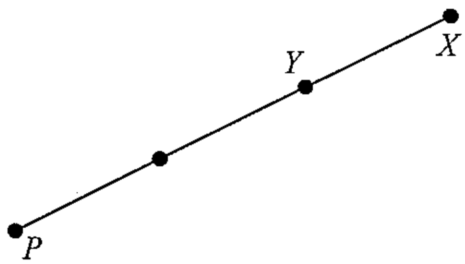
B)



C)



D)



ii) The curve $y = f(x)$ is defined for all values of x.
Given that $a < b < c$ and K is a constant, which one of the following is NOT necessarily always true.

A) $\int_b^a f(x)dx = -\int_a^b f(x)dx$

B) $\int_c^a K\,f(x)dx = K\int_c^a f(x)dx$

C) $\int_c^a f(x)dx = \int_b^a f(x)dx + \int_c^b f(x)dx$

D) $\int_{-a}^a f(x)dx = 2\times\int_0^a f(x)dx$

iii) If $y = f(x)$ is defined at $x = a$. Which one of the following gives the necessary condition(s) for the curve $y = f(x)$ to be continuous at $x = a$.

A) $f\,'(a) = 0$

B) $f(a) = f'(a)$

C) $\lim\limits_{x \to -a} f(x) = f(a) = \lim\limits_{x \to a} f(x)$

D) $\lim\limits_{x \to a} f(x) = f\,'(a)$

iv) A parabola has parametric equations $x = 2p,\ y = -2 - 2p^2$. This parabola has :

A) Vertex $(0, -2)$ and Focus $(0, -2\frac{1}{2})$

B) Vertex $(0, 2)$ and Focus $(0, -1\frac{1}{2})$

C) Vertex $(0, -2)$ and Focus $(0, -1\frac{1}{2})$

D) Vertex $(0, 2)$ and Focus $(0, -2\frac{1}{2})$

v) The hyperbola $y = \dfrac{2x-3}{x+6}$ has a :

A) vertical asymptote with equation $x = -6$ and a horizontal asymptote with equation $y = 1\frac{1}{2}$

B) vertical asymptote with equation $x = -6$ and a horizontal asymptote with equation $y = 2$

C) vertical asymptote with equation $x = -6$ and a horizontal asymptote with equation $y = 0$

D) vertical asymptote with equation $x = -6$ and a horizontal asymptote with equation $y = -\frac{1}{2}$

vi) A student uses the substitution $y = \sqrt{x}$ to find $\int \dfrac{dx}{\sqrt{x(1-x)}}$.

Which one of the following represents a equivalent definite integral.

A) $2\int \dfrac{dy}{\sqrt{1-y^2}}$

B) $\dfrac{1}{2}\int \dfrac{dy}{\sqrt{1-y^2}}$

C) $2\int \dfrac{y\,dy}{\sqrt{1-y^2}}$

D) $\dfrac{1}{2}\int \dfrac{y\,dy}{\sqrt{1-y^2}}$

QUESTION 2 **9 marks** ***(start a new page)***

a) The point P $(-1,-3)$ divides the interval joining A $(5, 9)$ and B $(-3,-7)$ internally in the ratio $k:1$. Find k (3)

b) Solve $\dfrac{4x+3}{2x-1}<1$ (3)

c) Find the acute angle between the two curves $y=x^3$ and $y=(2x-1)^2$ at their point of intersection $(1, 1)$. Give your answer correct to the nearest degree. (3)

QUESTION 3 **9 marks** ***(start a new page)***

a) The equation $2x^3-9x^2+12x-4\cdot 01=0$ has a root near $x=2$

(i) If $x=2$ is chosen as the initial approximation, explain why Newton's Method for finding an improved approximation fails. (2)

(ii) Use Newton's Method once and $x=1\cdot 5$ as an initial approximation to the root of $2x^3-9x^2+12x-4\cdot 01=0$ to find an improved approximation to the root. (write your answer correct to 2 d.p.) (2)

b) (i) Show that the quadratic polynomial $3x^2+2x+1$ is positive definite. (1)

(ii) If $P(x)=x^3+x^2+x-8$, use the sign derivative of $P(x)$ to explain why the equation $P(x)=0$ has only one root for all real values of x. (1)

(iii) Given that this root lies between $x=1\cdot 5$ and $x=2$,

use the Halving the Interval Method for approximating a root once

to show that the root lies more accurately between $x=1\cdot 5$ and $x=1\cdot 75$ (2)

(iv) Is the root closer to $x=1\cdot 5$ or $x=1\cdot 75$. Give reasons for your answer. (1)

QUESTION 4 **9 marks** ***(start a new page)***

a) Show, by process of mathematical induction, that $3^{4n} - 1$ is a multiple of 80 for all positive integral values of n (3)

b) Use mathematical induction to prove that

$1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \ldots\ldots n\times 2^{n-1} = 1 + (n-1)\times 2^n$ for all positive integral values of n (4)

c) A student is asked to use Mathematical Induction to prove that $5^n > 3^n + 4^n$ for integers $n \geq 3$.

His incomplete solution to the proof is given below.

Complete Step 3 of the proof on your own paper (2)

Step 1 Prove true for $n = 1$

L.H.S. $= 5^n$ R.H.S. $= 3^n + 4^n$

$= 5^3$ when $n = 3$ $= 3^3 + 4^3$

$= 125$ $= 91$

$\therefore$ L.H.S. > R.H.S. for $n = 3$

$\therefore$ $5^n > 3^n + 4^n$ for $n = 3$

Step 2 Assume true for $n = k$

i.e. assume $5^k > 3^k + 4^k$

Step 3 Prove true for $n = k+1$, if true for $n = k$

i.e. prove $5^{k+1} > 3^{k+1} + 4^{k+1}$

....

....

....

....

Step 4 Since true for $n = 1$ ....

QUESTION 5 **9 marks** ***(start a new page)***

a) The normal at the point $P(4t, 2t^2)$ on the parabola $x^2 = 8y$ intersects the y axis at Q.

(i) Show that the gradient of the normal at P is $-\frac{1}{t}$ (1)

(ii) Show that the equation of the normal at P is $x + ty = 4t + 2t^3$ (1)

(iii) Find, in terms of t, the coordinates of Q. (1)

(iv) State the **parametric equations** of the locus of the midpoint of PQ. (1)

b) The points $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola $x^2 = 4y$.
The chord PQ ,when produced, always passes through the point $A(2,0)$.

(i) Noting that $p \neq q$, show that $p + q = pq$. (2)

(ii) Find the co-ordinates of M, the midpoint of PQ (1)

(iii) Find the **cartesian equation** of the locus of M (2)

QUESTION 6 **9 marks** ***(start a new page)***

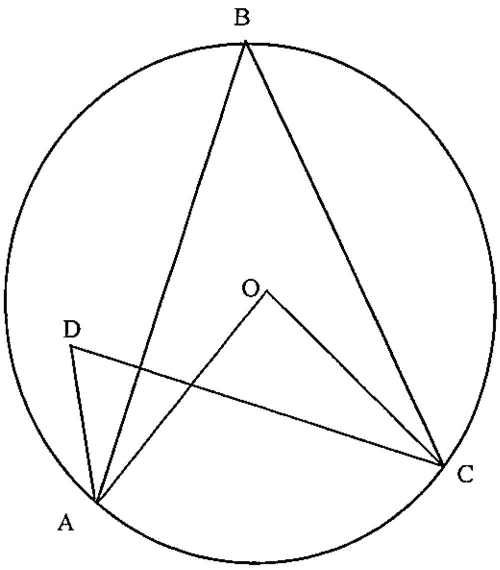
a) Evaluate the definite integral $\int \frac{t\,dt}{\sqrt{4-t}}$ using the substitution $t = 4 - w^2$ (3)

b) Find $\int \frac{x\,dx}{\sqrt{1-9x^2}}$ using the substitution $u = \sqrt{1-9x^2}$ (3)

c) Use the substitution $u = x + 1$ to evaluate $\int_0^3 \frac{x-2}{\sqrt{x+1}}\,dx$ (3)

QUESTION 7 **9 marks** ***(start a new page)***

a)

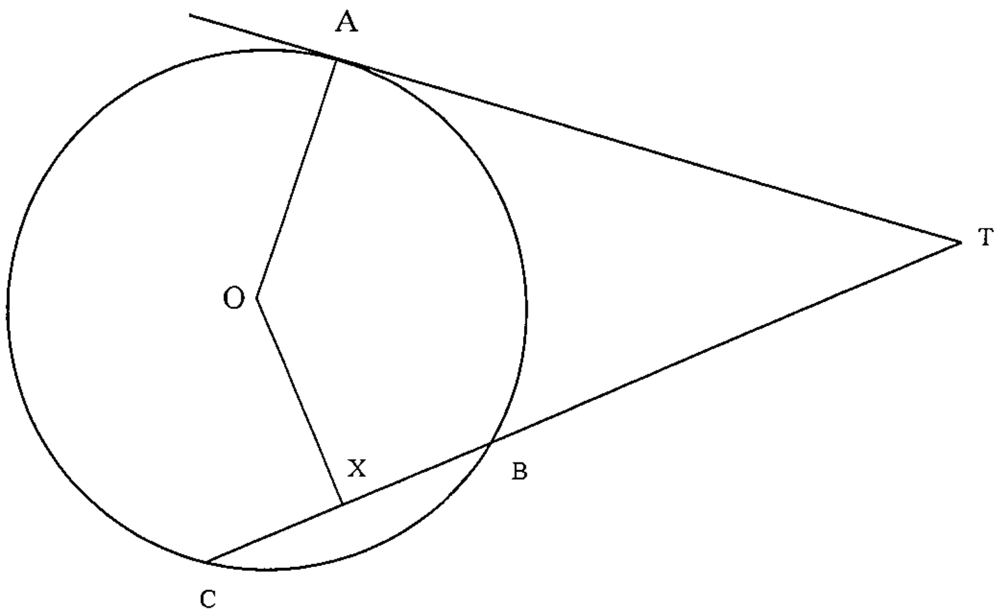


Points A, B and C lie on the circumference of a circle with centre O.
The point D lies inside the circle. $\angle ABC = 17°$ and $\angle ADC = 34°$

(i) Find $\angle AOC$ giving reasons. (1)

(ii) State why $ADOC$ is a cyclic quadrilateral. (1)

b)

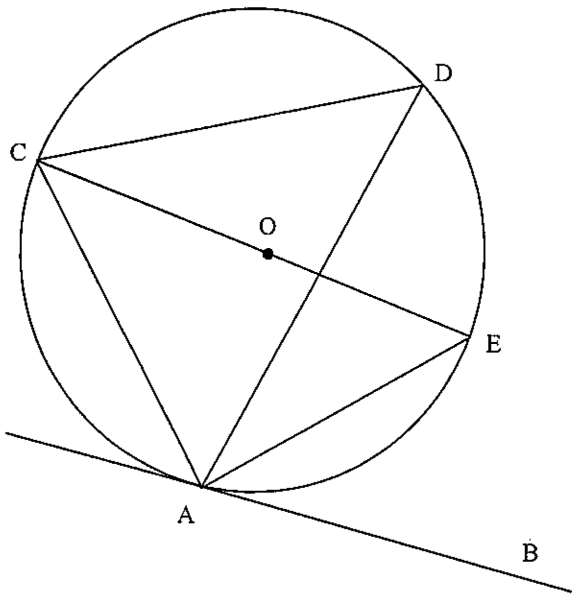


A, B and C are three points on a circle with centre O.

The tangent at A meets the chord CB produced at T. X is the midpoint of CB.

Prove that AOXT is a cyclic quadrilateral. (3)

c)



AB is a tangent and CE is a diameter to a circle with centre O.

$\angle BAE$ is $48°$ and D lies on the circumference as show in the diagram.

(i) Find the size of $\angle ACE$, giving reasons. (1)

(ii) Find the size of $\angle ADC$, justifying your answer with reasoning. (3)

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

© Board of Studies NSW 2009

GHS HSC MATHEMATICS EXTENSION 1 ASSESSMENT 2

SOLUTIONS

Question 1

(i) C (ii) D (iii) C

(iv) A (v) B (vi) A

Question 2 a) $A(5, 9)$ $B(-3, -7)$

$k : 1$

$$\therefore \frac{-3k+5}{k+1} = -1$$

$$-3k + 5 = -k - 1$$

$$6 = 2k$$

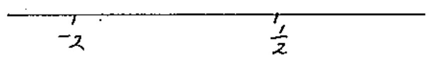
$$\therefore k = 3$$

b) Critical values occur when

$2x - 1 = 0$
$x = \frac{1}{2}$

and when $\frac{4x+3}{2x-1} = 1$

$$4x + 3 = 2x - 1$$
$$2x = -4$$
$$x = -2$$



FALSE when $x < -2$ (say $x = -3$)
TRUE when $-2 < x < \frac{1}{2}$ (say $x = 0$)
FALSE when $x > \frac{1}{2}$ (say $x = 1$)

$\therefore$ Solution is $-2 < x < \frac{1}{2}$

c) $y = x^3$ $y = (2x-1)^2$

$\frac{dy}{dx} = 3x^2$ $\frac{dy}{dx} = 2(2x-1)^1 \cdot 2$

$m_1 = 3$ when $x = 1$ $m_2 = 4$ when $x = 1$

Let θ be the required acute $\angle$.

$$\therefore \tan\theta = \left| \frac{m_2 - m_1}{1 + m_2 m_1} \right|$$

$$\tan\theta = \left| \frac{4 - 3}{1 + (4)(3)} \right|$$

$$\tan\theta = \frac{1}{13}$$

$$\theta = 4^\circ \text{ (to the nearest degree)}$$

Question 3 (a) (i) Let $P(x) = 2x^3 - 9x^2 + 12x - 4 \cdot 01$

$$P'(x) = 6x^2 - 18x + 12$$
$$P'(2) = 24 - 36 + 12$$
$$= 0$$

$\therefore$ Stationary Pt. exists at $x = 2$

$\therefore$ Tangent at $x = 2$ does not cross the x axis

The x-intercept of the tangent is the improved approximation. Thus an improved approximation cannot be found.

(ii) If $x_1 = 1 \cdot 5$ is the initial approx.

$P(x_1) = P(1 \cdot 5)$
$= 0 \cdot 49$

$P'(x_1) = P'(1 \cdot 5)$
$= -1 \cdot 5$

If x_2 is improved approx. then $x_2 = x_1 - \frac{P(x_1)}{P'(x_1)}$

$$= 1 \cdot 5 + \frac{0 \cdot 49}{1 \cdot 5}$$

$$= 1 \cdot 83 \text{ (to 2d.p)}$$

b) (i) $\Delta = b^2 - 4ac$
$= 4 - 4(3)(1)$
$= -8 < 0$

Since $a = 3 > 0$ and $\Delta < 0$ expression is +ve definite

(ii) $P'(x) = 3x^2 + 2x + 1 > 0$
∴ curve is always increasing
Since $P(x)$ is continuous for all x
the equation $P(x) = 0$ has only one root

(iii) $P(1.5) = -0.875 < 0$ $P(2) = 6$

Let $x_0 = \frac{1.5+2}{2}$ be new approx.

$x_0 = 1.75$

$P(1.75) = 2.171875 > 0$
$P(1.75)$ and $P(1.5)$ are opposite in sign
∴ Root lies between $x = 1.5$ and $x = 1.75$

(iv) Consider $P\left(\frac{1.5+1.75}{2}\right) = P(1.625)$
$= 0.55 > 0$

∴ Root lies between $x = 1.625$ and $x = 1.5$
∴ Root is closer $x = 1.5$ than $x = 1.75$

Question 4

a) Prove true for $n = 1$
$3^{4n} - 1 = 3^4 - 1$ when $n = 1$
$= 80$
Which is a multiple of 80
∴ true for $n = 1$

Assume true for $n = k$
ie $3^{4k} - 1 = 80M$ (where M is a +ve integer)
$3^{4k} = 80M + 1$

Prove true for $n = k+1$, if true for $n = k$
ie Prove $3^{4(k+1)} - 1$ is a multiple of 80
$3^{4(k+1)} - 1 = 3^{4k+4} - 1$
$= 3^4 \cdot 3^{4k} - 1$
$= 81(80M+1) - 1$ using assumption
$= 81 \times 80M + 81 - 1$
$= 80(81M + 1)$
Which is a multiple of 80 since $(81M+1)$ is a +ve integer

∴ true for $n = k+1$, if true for $n = k$

Since true for $n = 1$ ∴ true for $n = 1+1 = 2$
Since true for $n = 2$ ∴ true for $n = 2+1 = 3$

and so on.

∴ true for all given 'n'

Question 4 (b) Prove true for $n=1$

$$L.H.S. = n \times 2^{n-1} = 1 \times 2^0 \text{ when } n=1 = 1$$

$$R.H.S = 1 + (n-1) \times 2^n = 1 + 0 \text{ when } n=1 = 1 = L.H.S$$

$\therefore$ true for $n=1$

Assume true for $n=k$

ie $1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \cdots + k \times 2^{k-1} = 1 + (k-1) \times 2^k$

Prove true for $n=k+1$, if true for $n=k$

i.e. Prove $1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \cdots + k \times 2^{k-1} + (k+1) \times 2^k = 1 + k \times 2^{k+1}$

$$\begin{aligned} L.H.S. &= 1 + (k-1) \times 2^k + (k+1) \times 2^k \quad \text{using assumption} \\ &= 1 + 2^k (k-1+k+1) \\ &= 1 + 2^k \times 2k \\ &= 1 + k \times 2^{k+1} \\ &= R.HS. \end{aligned}$$

$\therefore$ true for $n=k+1$ if true for $n=k$

Since true for $n=1$ $\therefore$ true for $n=1+1=2$

Since true for $n=2$ $\therefore$ true for $n=2+1=3$

and so on

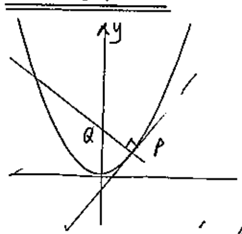
$\therefore$ true for all give 'n'

Question 4 c) $5^{k+1} = 5 \times 5^k$

$$\begin{aligned} \therefore 5^{k+1} &> 5(3^k + 4^k) \quad \text{using assumption} \\ 5^{k+1} &> 5 \times 3^k + 5 \times 4^k \\ 5^{k+1} &> 3 \times 3^k + 4 \times 4^k \quad \text{since } 3<5 \text{ and } 4<5 \\ 5^{k+1} &> 3^{k+1} + 4^{k+1} \end{aligned}$$

$\therefore$ true for $n=k+1$ if true for $n=k$

Question 5



a) (i) $y = \frac{1}{8}x^2$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{4}x \\ &= \frac{1}{4} \times 4t \quad \text{at } P \\ &= t \quad \text{(gradient of tangent at } P) \end{aligned}$$

$\therefore$ Gradient of normal at P is $-\frac{1}{t}$

(ii) Equation of normal is $y - 2t^2 = -\frac{1}{t}(x - 4t)$

i.e $ty - 2t^3 = -x + 4t$

$x + ty = 4t + 2t^3$

(iii) At y axis $x = 0$

$\therefore ty = 4t + 2t^3$

$y = 4 + 2t^2$

$\therefore Q(0, 4+2t^2)$

(iv) Midpoint of PQ is $\left(\frac{4t}{2}, \frac{2t^2+4+2t^2}{2}\right)$

$= (2t, 2t^2+2$

$\therefore$ Parametric equations are $x = 2t$, $y = 2t^2 + 2$

b) (i) $m_{AP} = m_{AQ}$

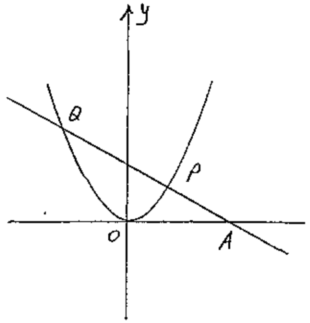
$$\frac{p^2}{2p-2} = \frac{q^2}{2q-2}$$

$$2p^2q - 2p^2 = 2q^2p - 2q^2$$

$$2p^2q - 2q^2p = 2p^2 - 2q^2$$

$$2pq(p-q) = 2(p-q)(p+q)$$

$$pq = p+q \quad \text{since } p \neq q$$



(ii) M is $\left(\frac{2p+2q}{2}, \frac{p^2+q^2}{2}\right)$

$$= \left(p+q, \frac{p^2+q^2}{2}\right)$$

(iii) Parametric equations are

$x = p+q$ and $y = (p+q)^2 - 2pq$.

$\therefore y = x^2 - 2(p+q)$ using (i)

$y = x^2 - 2x$ is the cartesian equation

Question 6 a) $t = 4 - w^2 \rightarrow w^2 = 4 - t$

$$\frac{dt}{dw} = -2w$$

$$\therefore \int \frac{t\,dt}{\sqrt{4-t}} = \int \frac{4-w^2}{\sqrt{w^2}} \times -2w \; dw$$

$$= 2\int w^2 - 4 \; dw$$

$$= 2\left[\frac{w^3}{3} - 4w\right] + c$$

$$= \frac{2}{3}\sqrt{(4-t)^3} - 8\sqrt{4-t} + c$$

b) $u = (1-9x^2)^{\frac{1}{2}}$

$$\frac{du}{dx} = \frac{1}{2}(1-9x^2)^{-\frac{1}{2}} \times -18x$$

$$= \frac{-9x}{\sqrt{1-9x^2}}$$

$$\frac{dx}{du} = \frac{u}{-9x}$$

$$\therefore \int \frac{x\,dx}{\sqrt{1-9x^2}} = \int \frac{x}{u} \times \frac{u}{-9x} \; du$$

$$= -\frac{1}{9}\int 1 \; du$$

$$= -\frac{1}{9}u + c$$

$$= -\frac{1}{9}\sqrt{1-9x^2} + c$$

Question 6 c)

$u = x+1$ when $x = 3$, $u = 4$

$\frac{du}{dx} = 1 \rightarrow \frac{dx}{du} = 1$ $x = 0$, $u = 1$

$$\therefore \int_0^3 \frac{x-2}{\sqrt{x+1}}\,dx = \int_1^4 \frac{u-3}{\sqrt{u}} \times 1\,du$$

$$= \int_1^4 \frac{u}{u^{\frac{1}{2}}} - \frac{3}{u^{\frac{1}{2}}}\,du$$

$$= \int_1^4 u^{\frac{1}{2}} - 3u^{-\frac{1}{2}}$$

$$= \left[\frac{2u^{\frac{3}{2}}}{3} - 6u^{\frac{1}{2}}\right]_1^4$$

$$= \left[\frac{2}{3}\times 8 - 6\times 2 - \left(\frac{2}{3} - 6\right)\right]$$

$$= \frac{14}{3} - 6$$

$$= -\frac{4}{3}$$

Question 7 a)

(i) $\angle AOC = 2 \times \angle ABC$ ($\angle$. at the centre is twice the angle at the circumference, standing on the same arc)

$= 2 \times 17^\circ$

$= 34^\circ$

(ii) $\angle AOC = \angle ADC$

$= 34^\circ$

Since the interval AC subtends two equal $\angle$'s at O and D then the end points of the interval, A & C and the vertices of the angles, O & D are concyclic i.e ADOC is a cyclic quadrilateral

Question 7

b) $\angle AOT = 90^\circ$ (tangent is perpendicular to the radius drawn at the point of contact)

$\angle AXB = 90^\circ$ (interval drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord)

$\therefore \angle AOT + \angle AXB = 180^\circ$

Since the opposite angles of the quadrilateral AOXT are supplement, AOXT must be a cyclic quadrilateral.

c) (i) $\angle ACE = \angle ABE$ ($\angle$. between a tangent and a chord drawn at the point of contact is equal to the angle in the alternate segment)

$= 48^\circ$

(ii) $\angle CAE = 90^\circ$ (angle in a semi-circle is a right angle)

$\angle CEA = 180 - (90+48)^\circ$ ($\angle$. sum of ΔACE)

$= 42^\circ$

$\angle ADC = \angle CEA$ (angles at the circumference standing on the same arc are equal)