Name________________________ Teacher____________________

GOSFORD HIGH SCHOOL

2013

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 2

MATHEMATICS – EXTENSION 1

Duration- 90 minutes plus 5 minutes reading time

Section 1 **Multiple choice**	5 questions worth **1 mark each**. (Answer this section on the **multiple choice response sheet provided)**	/5
Section 2 **Question 6**	Angle between 2 lines, Iterative methods, Integration	/15
Question 7	Locus, Integration	/15
Question 8	Integration, Mathematical Induction, Graphing	/15
TOTAL		**/50**

SECTION 1: MULTIPLE CHOICE

Questions 1 – 5 are to be answered on the multiple choice answer sheet provided. Each question is worth 1 mark.

Question 1. The exact value of $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$ is:

A) $\frac{\pi}{8}+\frac{1}{4}$ **B)** $\frac{\pi}{4}-\frac{1}{2}$

C) $\frac{\pi}{8}-\frac{1}{4}$ **D)** $\frac{\pi}{4}+\frac{1}{2}$

Question 2. Using the substitution $u=1-x^3$, $\int \frac{x^2}{\sqrt{1-x^3}}dx$ is equivalent to:

A) $-3\int \frac{du}{\sqrt{u}}$ **B)** $\frac{-1}{3}\int \frac{du}{\sqrt{u}}$

C) $\frac{-1}{3}\int \frac{du}{u^2}$ **D)** $3\int \frac{du}{u^2}$

Question 3. The curve $y=\frac{(2x+3)(x-2)}{(x-3)(x+1)}$, has:

A) Vertical asymptotes at $x=3$ and $x=-1$ and a horizontal asymptote at $y=2$.

B) Vertical asymptotes at $x=-3$ and $x=1$ and a horizontal asymptote at $y=2$.

C) Vertical asymptotes at $x=3$ and $x=-1$ and horizontal asymptotes at $y=-\frac{3}{2}$ and $y=2$.

D) Vertical asymptotes at $x=3$ and $x=-1$ and a horizontal asymptote at $y=0$.

Question 4. The parabola with parametric equations $x = 2t + 1$ and $y = t^2 - 2$, has:

A) Vertex (-1, 2) and focus (-1, 3)

B) Vertex (-1, 2) and focus (0, 3)

C) Vertex (1, -2) and focus (1, -3)

D) Vertex (1, -2) and focus (1, -1)

Question 5. When using mathematical induction to prove

$1 + 3 + 3^2 + \ldots\ldots + 3^{n-1} = \dfrac{3^n - 1}{2}$, step 3 would read:

Prove true for $n = k + 1$, if true for $n = k$, ie prove:

A) $\dfrac{3^k - 1}{2} + 3^{k+1} = \dfrac{3^{k+1} - 1}{2}$

B) $\dfrac{3^k - 1}{2} + 3^k = \dfrac{3^{k+1} - 1}{2}$

C) $\dfrac{3^k}{2} + 3^{k+1} = \dfrac{3^{k+1} - 1}{2}$

D) $\dfrac{3^k - 2 + 3^{k+1}}{2} = \dfrac{3^{k+1} - 1}{2}$

SECTION 2: FREE RESPONSE

Questions 6,7 and 8 are to be answered on your own paper. Start each question on a new page. All necessary working must be shown. Marks may not be awarded for untidy or poorly set out work.

Question 6.

a) Find the acute angle between the lines $y = 3x - 2$ and $y = 2 - x$, giving your answer to the nearest degree. **(2)**

b) Find the acute angle between the curves $y = 3x^3 - 2$ and $y = (x-2)^2$ at their point of intersection (1,1). Give answer correct to the nearest degree. **(3)**

c) Let $f(x) = x^3 + 5x^2 + 17x - 10$. The equation $f(x) = 0$ has only one real root. Given that the root lies between 0 and 2:

i) Use one application of the 'halving the interval' method to find a smaller interval containing the root. **(2)**

ii) Which end of the interval found in part i) is closer to the root? Justify your answer. **(2)**

d) Use Newton's method to find a second approximation to the positive root of $x - 2\sin x = 0$. Take $x = 1.7$ as the first approximation. Give answer correct to 2 decimal places. **(3)**

e) Evaluate $\int_0^1 6x\sqrt{9 - x^2}\,dx$, using the substitution $u = 9 - x^2$. Leave your answer in surd form. **(3)**

Question 7. **Start a new page.**

a) Find the equation of the chord of contact of the tangents to the parabola $x^2 = y$ from the point (-1, -5). **(3)**

b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The equation of the tangents at P and Q respectively are
$y = px - ap^2$ and $y = qx - aq^2$.

i) The tangents at P and Q meet at the point T. Show that the coordinates of T are $(a(p+q), apq)$. **(2)**

ii) Find the equation of the locus of T if PQ is a focal chord. **(2)**

c) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$. The chord PQ subtends a right angle at the origin. M is the mid point of PQ.

i) Show that $pq = -4$ **(2)**

ii) Show that the locus of M is given by $x^2 = 2a(y - 4a)$ **(3)**

d) Find $\int x\sqrt{x+1}\,dx$, using the substitution $u^2 = x + 1$. **(3)**

Question 8. Start a new page.

a) The region under the curve $y = \cos x + \sec x$, above the x axis and between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x axis. Show that the volume of the solid formed is $\frac{5\pi(\pi+2)}{8}$ cubic units. **(4)**

b) Prove by mathematical induction that

$3^n \geq 1 + 2n$ for all integers n, $n \geq 1$. **(4)**

c) Consider the function $f(x) = \dfrac{x^4 + 3x^2}{x^4 + 3}$.

i) Show that $f(x)$ is an even function. **(1)**

ii) What is the equation of the horizontal asymptote to the graph $y = f(x)$? **(1)**

iii) Find the x coordinates of all stationary points for the graph $y = f(x)$? **(3)**

iv) Sketch the graph $y = f(x)$. You are not required to find any points of inflexion. **(2)**

HSC EXT 1 ASSESSMENT 2 2013

SOLUTIONS

SECTION 1: Multiple choice

1) $$\int_0^{\pi/4} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/4} 1 - \cos 2x\,dx$$
$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/4}$$
$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right]$$
$$= \frac{\pi}{8} - \frac{1}{4}$$
———— (C)

2) $$\int \frac{x^2}{\sqrt{1-x^3}}\,dx = -\frac{1}{3}\int \frac{-3x^2}{\sqrt{1-x^3}}\,dx$$
$u = 1 - x^3$

$\frac{du}{dx} = -3x^2$

$$= -\frac{1}{3}\int \frac{du}{\sqrt{u}}$$
———— (B)

3) $$y = \frac{(2x+3)(x-2)}{(x-3)(x+1)}$$

$$\lim_{x\to\infty} \frac{2x^2 - x - 6}{x^2 - 2x - 3}$$
$$= \lim_{x\to\infty} \frac{\frac{2x^2}{x^2} - \frac{x}{x^2} - \frac{6}{x^2}}{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{3}{x^2}}$$
$$= 2$$
ie horizontal asymptote $y = 2$

$(x-3)(x+1) \neq 0$

$\therefore x \neq 3, -1$

ie vertical asymptotes $x = 3$ and $x = -1$

———— (A)

4) $x = 2t+1$, $y = t^2 - 2$

$2t = x-1$

$t = \frac{x-1}{2}$

$y = \left(\frac{x-1}{2}\right)^2 - 2$

$4y = (x-1)^2 - 8$

$(x-1)^2 = 4y+8$

$(x-1)^2 = 4(y+2)$

$\therefore$ vertex $(1,-2)$ —— (D)

focus $(1,-1)$

5) $1+3+3^2+\ldots+3^{k-1}+3^k = \frac{3^{k+1}-1}{2}$

$\frac{3^k-1}{2} + 3^k = \frac{3^{k+1}-1}{2}$ —— (B)

SECTION 2

6) a) $y = 3x-2$ $\quad$ $y = 2-x$

$M_1 = 3$ $\quad$ $M_2 = -1$

$\tan\theta = \left|\frac{M_1 - M_2}{1+M_1M_2}\right|$

$= \left|\frac{3+1}{1-3}\right|$

$= \left|-\frac{3}{2}\right|$

$\tan\theta = \frac{3}{2}$

$\theta = 56^\circ$

b) $y = 3x^3 - 2$ $\quad$ $y = (x-2)^2$

$y' = 9x^2$ $\quad$ $y' = 2(x-2)$

when $x=1$, $\quad$ when $x=1$,

$M_1 = 9$ $\quad$ $M_2 = -2$

$\tan\theta = \left|\frac{M_1 - M_2}{1+M_1M_2}\right|$

$= \left|\frac{9+2}{1-18}\right|$

$= \frac{11}{17}$

$\theta = 33^\circ$

c) $f(x) = x^3 + 5x^2 + 17x - 10$

i) $f(0) = -10 < 0$ $\quad$ $f(2) = 8+20+34-10$

$= 52 > 0$

$f(1) = 1+5+17-10$

$= 13 > 0$ $\quad$ $\therefore$ root lies between 0 and 1 (i.e $0 < x < 1$)

should include 'since there is a sign change from 0 to 1, and it is continuous.'

ii) $f\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{5}{4} + \frac{17}{2} - 10$

$= -\frac{1}{8}$ $\quad$ $\therefore$ root lies between $\frac{1}{2}$ and 1

$\therefore$ 1 is closer to the root than zero.

d) $f(x) = x - 2\sin x$ $\qquad f(1.7) = 1.7 - 2\sin 1.7$

$= -0.2833$

$f'(x) = 1 - 2\cos x$

$f'(1.7) = 1 - 2\cos 1.7$

$= 1.2577$

$$a_1 = a - \frac{f(a)}{f'(a)}$$

$$= 1.7 + \frac{0.2833}{1.2577}$$

$$= 1.93$$

e) $\displaystyle\int_0^1 6x\sqrt{9-x^2}\,dx$

let $u = 9 - x^2$ $\qquad$ when $x=0$, $u=9$

$\frac{du}{dx} = -2x$ $\qquad$ $x=1$, $u=8$

$du = -2x\,dx$

$$-3\int_0^1 -2x\sqrt{9-x^2}\,dx = -3\int_9^8 u^{1/2}\,du$$

$$= -3\left[\frac{2u^{3/2}}{3}\right]_9^8$$

$$= -2\left[8^{3/2} - 9^{3/2}\right]$$

$$= -2\left[16\sqrt{2} - 27\right]$$

7. a) $x^2 = y$ $\qquad (-1, -5)$

$xx_1 = 2a(y + y_1)$

$x.-1 = 2.\frac{1}{4}(y-5)$

$-x = \frac{1}{2}(y-5)$

$-2x = y - 5$

$y = -2x + 5$

b) $P(2ap, ap^2)$, $Q(2aq, aq^2)$ $\quad x^2 = 4ay$

i) $px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p-q)(p+q)$

$x = a(p+q)$

sub into $y = px - ap^2$

$y = ap(p+q) - ap^2$

$= ap^2 + apq - ap^2$

$= apq$

$\therefore$ T has coordinates $(a(p+q), apq)$

ii) If PQ is a focal chord, $pq = -1$

$\therefore y = apq$

$y = -a$ $\qquad$ ie locus is the directrix

c) $P(2ap, ap^2)$ $Q(2aq, aq^2)$ $x^2 = 4ay$

i) If PQ subtends a right angle at the origin, then

$$M_{OP} \cdot M_{OQ} = -1$$

$$M_{OP} = \frac{ap^2}{2ap} = \frac{p}{2} \qquad M_{OQ} = \frac{aq^2}{2aq} = \frac{q}{2}$$

$$\therefore \quad \frac{p}{2} \cdot \frac{q}{2} = -1$$
$$pq = -4$$

ii) Midpoint

$$x = \frac{2ap+2aq}{2} = \frac{a(p+q)}{2} \qquad y = \frac{aq^2+ap^2}{2} = \frac{a(q^2+p^2)}{2}$$

$$x^2 = a^2(p+q)^2$$
$$= a^2(p^2+q^2+2pq)$$
$$= a^2(p^2+q^2-8)$$
$$\frac{x^2}{a^2} = p^2+q^2-8$$
$$p^2+q^2 = \frac{x^2}{a^2}+8$$

now $y = \frac{a(q^2+p^2)}{2}$

$$= \frac{a\left(\frac{x^2}{a^2}+8\right)}{2}$$
$$2y = \frac{x^2}{a}+8a$$
$$x^2 = 2ay - 8a^2$$
$$x^2 = 2a(y-4a)$$

d) $\int x\sqrt{x+1}\,dx$

let $u^2 = x+1$

$$x = u^2 - 1$$
$$\frac{dx}{du} = 2u$$
$$dx = 2u\,du$$

$$\int x\sqrt{x+1}\,dx = \int (u^2-1)u \cdot 2u\,du$$
$$= \int 2u^4 - 2u^2\,du$$
$$= 2\left[\frac{u^5}{5} - \frac{u^3}{3}\right]$$
$$= 2\left[\frac{(\sqrt{x+1})^5}{5} - \frac{(\sqrt{x+1})^3}{3}\right] + c$$

8) a) $y = \cos x + \sec x$

$$V = \pi\int_0^{\pi/4} (\cos x + \sec x)^2\,dx$$
$$= \pi\int_0^{\pi/4} \cos^2 x + 2\cos x \sec x + \sec^2 x\,dx$$
$$= \pi\int_0^{\pi/4} \cos^2 x + 2 + \sec^2 x\,dx$$
$$= \pi\int_0^{\pi/4} \frac{1}{2}(1+\cos 2x) + \sec^2 x + 2\,dx$$

$$= \pi \left[\frac{1}{2}x + \frac{1}{4}\sin 2x + \tan x + 2x \right]_0^{\pi/4}$$

$$= \pi \left[\frac{\pi}{8} + \frac{1}{4}\sin\frac{\pi}{2} + \tan\frac{\pi}{4} + \frac{\pi}{2} - 0 \right]$$

$$= \pi \left[\frac{\pi}{8} + \frac{1}{4} + 1 + \frac{\pi}{2} \right]$$

$$= \pi \left[\frac{5\pi + 10}{8} \right]$$

$$= \frac{5\pi(\pi+2)}{8} \text{ cubic units}$$

b) Prove $3^n \geq 1+2n$, $n \geq 1$

Step 1: Show true for $n=1$

$LHS = 3^1 = 3$ $\quad$ $RHS = 1+2 = 3$

$LHS \geq RHS$

$\therefore$ true for $n=1$

Step 2: Assume true for $n=k$, k an integer

ie $3^k \geq 1+2k$

Step 3: Prove true for $n=k+1$, if true for $n=k$

ie Prove $3^{k+1} \geq 1+2(k+1)$

$3^{k+1} \geq 3+2k$

now $3^{k+1} = 3.3^k$

and $3.3^k \geq 3(1+2k)$ from assumption

$3^{k+1} \geq 3+6k$

now for $k \geq 1$, $6k > 2k$

$\therefore 3^{k+1} \geq 3+2k$

$\therefore$ true for $n=k+1$, if true for $n=k$

$\therefore$ proven true by mathematical induction

c) $f(x) = \dfrac{x^4+3x^2}{x^4+3}$

i) $f(-x) = \dfrac{(-x)^4+3(-x)^2}{(-x)^4+3}$

$= \dfrac{x^4+3x^2}{x^4+3}$

$= f(x)$ $\quad \therefore$ even fn.

ii) $\lim\limits_{x \to \infty} \dfrac{\frac{x^4}{x^4} + \frac{3x^2}{x^4}}{\frac{x^4}{x^4} + \frac{3}{x^4}}$

$= \lim\limits_{x \to \infty} \dfrac{1 + \frac{3}{x^2}}{1 + \frac{3}{x^4}}$

$= 1$ $\quad \therefore$ horizontal asymptote is $y=1$