Student Name/Number ......Mine......

ASSESSMENT TASK 2- March 2015

MATHEMATICS

EXTENSION 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 90 Minutes
- Start each question on a new page.
- All necessary working should be shown in every question.
- Topics tested-
- Qn 8-preliminary, integration by substitution
- Qn 9 Parabola, locus, Newton's method
- Qn 10- Induction, approximations, harder graphs

QUESTION	MARK
MC /7	
8 /15	
9 /15	
10 /15	
Total /52	

1. The expansion needed to show $\sin 75° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ is:

   (A) $\sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

   (B) $\sin(45° + 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

   (C) $\sin(100° - 25°) = \sin 100° \cos 25° + \cos 100° \sin 25°$

   (D) $\tan(45° + 30°) = \dfrac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

2. The interval joining the points $A(-3,2)$ and $B(-9,y)$ is divided externally in the ratio 5:3 by the point $P(x,-13)$. What are the values of x and y?

   (A) $x=-27,\ y=22$ (B) $x=27,\ y=4$

   (C) $x=6,\ y=12$ (D) $x=-18,\ y=-4$

3. Find the $\lim\limits_{x\to 0}\left(\dfrac{\sin x\cos x}{2x}\right)$

   (B) 2 (B) 1

   (C) $\dfrac{1}{2}$ (D) $\dfrac{1}{4}$

4. Which of the following is an expression for $\int \cos^2 2x\,dx$?

   (A) $x-\dfrac{1}{4}\sin 4x+C$ (B) $x+\dfrac{1}{4}\sin 4x+C$

   (C) $\dfrac{x}{2}-\dfrac{1}{8}\sin 4x+C$ (D) $\dfrac{x}{2}+\dfrac{1}{8}\sin 4x+C$

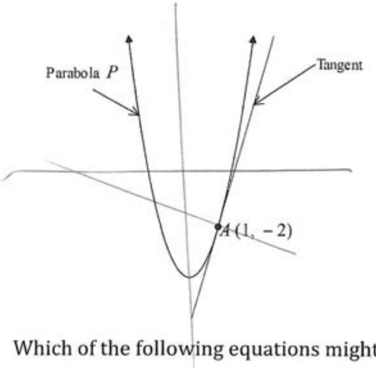
5. The Cartesian equation of the tangent, at $t = -3$, to the parabola

$x = t - 3,\ y = t^2 + 2$ is:

(A) $6x + y + 25 = 0$ (B) $6x + y + 36 = 0$

(C) $6x - y - 25 = 0$ (D) $6x + 2y - 25 = 0$

6. The diagram shows the parabola P and the tangent at the point $A\,(1, -2)$.



Which of the following equations might represent the normal to the parabola at A?

(A) $x - 3y + 5 = 0$ (B) $2x - 3y + 1 = 0$

(C) $x + 3y + 5 = 0$ (D) $x + 3y - 5 = 0$

7. If $f(x) = \sin^2(3 - x)$ then $f'(0) =$

(A) $-2\cos(3)$ (B) $-2\sin(3)\cos(3)$

(C) $2\sin(3)\cos(3)$ (D) $6\sin(3)\cos(3)$

QUESTION 8 (15 MARKS) Answer this question on a new page.

a. The graphs of $y = 8 - x^3$ and $x - 2y + 13 = 0$ intersect at the point $(1, 7)$. Find the size of the acute angle between the tangent to the curve and the line at the point of intersection. (answer to the nearest minute) 2

b. Use the substitution $u = 5 - x^2$ to evaluate 2

$$\int \frac{x}{(5 - x^2)^3}\, dx\,.$$

c. Use the substitution $u = x - 3$ to evaluate

$$\int_3^4 x\sqrt{x - 3}\, dx$$ 3

d. Solve $\dfrac{2x + 3}{x} \geq x$. 3

e. Find the exact value of 3

$$\int_{\frac{\pi}{2}}^{\pi} (\sin^2 x + x)\, dx\,.$$

f. Show that $x = 5\sin\theta$ and $y = 5\cos\theta + 1$ satisfies the equation 2

$$y^2 + x^2 - 2y - 24 = 0$$

QUESTION 9 (15 MARKS) Answer this question on a new page.

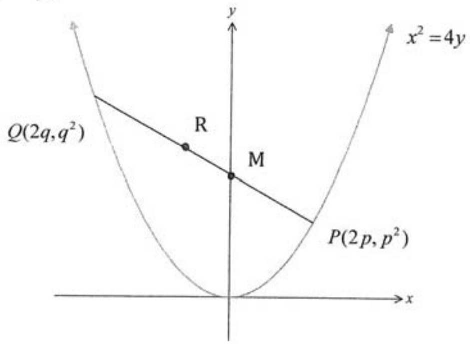
a) $P(2p,p^2)$ and $Q(2q,q^2)$ are two points on the parabola $x^2 = 4y$.

(i) Find the coordinates of M, the mid point of PQ. 1

(ii) Show $pq = -4$ if PQ subtends a right angle at the origin. 3

(iii) Using your answers to parts (i) and (ii), find the equation of the locus of M as P and Q move on the parabola if $\angle POQ = 90°$. 2

b) The diagram shows two distinct points $P(2p,p^2)$ and $Q(2q,q^2)$ that lie on the parabola $x^2 = 4y$.



The normal to the parabola at P intersects the y axis at M which is the midpoint of PR. If the equation of the normal is

$$x + py - 2p - p^3 = 0$$

(1) Find the co-ordinates of M 1

(2) The locus of the point R is a parabola. Find the equation of this parabola in Cartesian form and state its vertex. 3

c) i) Show that the equation $\cos x = x$ has a root lying between $x = 0.7$ and $x = 0.8$ 2

ii) Using $x = 0.75$ as a first approximation, use one application of Newton's Method to find a better approximation to 3 decimal places 2

iii) Draw a diagram to explain any situation where Newton's method fails 1

QUESTION 10 (15 MARKS) Answer this question on a new page.

a) Prove by mathematical induction that for any positive integer $n \geq 1$ 4

$$\frac{1}{1\times 5}+\frac{1}{5\times 9}+\frac{1}{9\times 13}+\dots\dots\dots+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}$$

b) Prove by mathematical induction that for any positive integer $n \geq 1$ 4

$12^n + 2\times 5^{n-1}$ is divisible by 7

c) Let $g(x) = 2x^3 + x + 4$

i. If $g(x) = 0$ has a root between the integers -1 and -2 and you are given $g(-1) = 1$ and $g(-2) = -14$, use halving the interval method once to 1

nearest integer

approximate a root, to1 decimal place.

ii. Using a second application of halving the interval method state the domain in which a better approximation would lie 1

iii. Explain why this function $y = g(x)$ has only one real root. 1

d) i. By sketching two appropriate graphs on the same number plane

solve $x+4 > \dfrac{2}{x+3}$ 3

ii. Hence or otherwise deduce the values of x for which $x+4 > \dfrac{2}{|x+3|}$ 1

END OF TASK

a) prove true for $n=1$

$LHS = \frac{1}{1\times 5} = \frac{1}{5}$ $\quad RHS = \frac{n}{4n+1} = \frac{1}{4+1}$

$\therefore LHS = RHS$

assume true for $n=k$

$\frac{1}{1\times5} + \frac{1}{5\times9} + \dots \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$

prove true for $n=k+1$

$\frac{1}{1\times5} + \dots \frac{1}{(4k-3)(4k+1)} + \frac{1}{(4(k+1)-3)(4(k+1)+1)}$

$= \frac{k}{4k+1} + \frac{1}{(4k+4-3)(4k+4+1)}$

$= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$

$= \frac{k(4k+5)+1}{(4k+1)(4k+5)}$

$= \frac{4k^2+5k+1}{(4k+1)(4k+5)}$

$= \frac{\cancel{(4k+1)}(k+1)}{\cancel{(4k+1)}(4k+5)} = \frac{k+1}{(4(k+1)+1)}$

since it is true for $n=1$ it is true for $n=1+1=2$.
The statement is true for $n=k+1$ if true for $n=k$

as $6x^2+1>0$
$\therefore$ fn only intersects x axis once

b) Prove true for $n=1$

$12^n + 2\times5^{n-1} = 12 + 2\times5^0$
$= 12+2$
$= 14 = 2\times7$

$\therefore$ true for $n=1$

step 2. Assume true for $n=k$

$12^k + 2\times5^{k-1} = 7M$ (M an integer)

$12^k = 7M - 2\times5^{k-1}$

step 3. Prove true for $n=k+1$ if true for $n=k$.

$12^{k+1} + 2\times5^k = 12\times2^k + 2\times5^k$

$= 12(7M - 2\times5^{k-1}) + 2\times5^k$

$= 12\times7M - 24\times5^{k-1} + 2\times5^k$

$= 12\times7M - 24\times5^{k-1} + 2\times5\times5^{k-1}$

$= 12\times7M - 14\times5^{k-1}$

$= 7(12M - 2\times5^{k-1})$

$= 7Q$ $\quad$ Q an integer

etc. . . .

c) $g\left(\frac{-1+-2}{2}\right) = g(-1\cdot5)$

$\therefore 2\times(-1\cdot5)^3 + (-1\cdot5) + 4$

$= -4\cdot25$ $\quad$ 1

$x \doteqdot -1$ from 1st approx

ii) use between $x=-1$ and $x=-1\cdot5$

use $g(-1\cdot25)$

$2(-1\cdot25)^3 + (-1\cdot25) + 4$

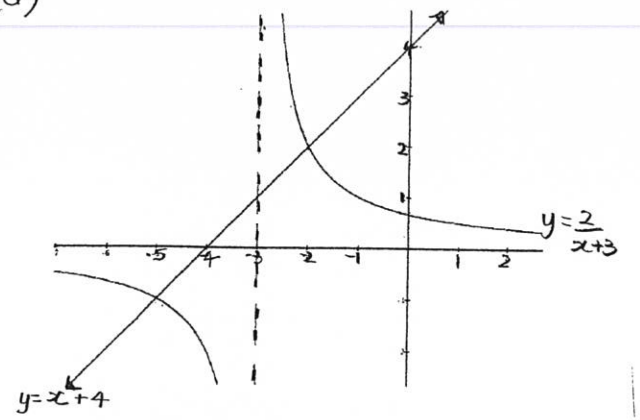
$= -1\cdot156$

$\therefore$ root lies between $x=-1$ and $x=-1\cdot25$

iii) $g'(x)$ always >0

$\therefore$ always increasing

d)



Find the points of intersection

$x+4 = \frac{2}{x+3}$

$(x+4)(x+3) = 2$

$x^2+7x+12 = 2$

$x^2+7x+10 = 0$

$(x+2)(x+5) = 0$

$x=-2 \quad x=-5$

$\therefore -5 < x < -3$ or $x > -2$

ii) $\frac{2}{|x+3|}$ means only where this curve is positive ie above x axis

$\therefore x > -2$

1 A

2. $(-3,2)$ $(-9,y)$ D

$5:-3$

$\frac{9+-45}{2}$, $\frac{-6+5y}{2}$

$x=\frac{-36}{2}$ $\quad -13=\frac{-6+5y}{2}$

$=-18$ $\quad -26=-6+5y$

$y=-4$

3. $\lim_{x\to 0} \frac{1}{2}\cdot\frac{2\sin x\cos x}{2x}$

$=\frac{1}{2}\frac{\sin 2x}{2x}$

$=\frac{1}{2}$ C

4. $\int \frac{1+\cos 4x}{2}\,dx$

$=\frac{1}{2}x+\frac{\sin 4x}{8}+C$ D

5. $x=t-3$ $\quad y=t^2+2$

$x+3=t$ $\quad y=x^2+6x+11$

$\frac{dy}{dx}=2x+6$

when $t=-3$

$x={}^-6$ $\quad y=11$

$y-11=-6(x+6)$ A

$y-11=-6x-36$

$6x+y+25=0$

6. C or D have neg grad.

C) $3y=-x-5$ $\quad$ D) $3y=-x+5$

sub $(1,-2)$ in

$-6=-1-5$ ✓ C

so C.

7. $f(x)=[\sin(3-x)]^2$

$f'(x)=2\sin(3-x)\cdot-\cos(3-x)$

$f'(0)=2\sin 3\cdot-\cos 3$

$=-2\sin 3\cos 3$ B.

QUESTION 8.

$y=8-x^3$ $\quad x+13=2y$

$\frac{dy}{dx}=-3x^2$ $\quad \frac{1}{2}x+\frac{13}{2}=y$

at $x=1$ $\quad m_2=\frac{1}{2}$

$m=-3$

$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$=\left|\frac{-3-\frac{1}{2}}{1+-3.\frac{1}{2}}\right|$

$=\left|\frac{-3\frac{1}{2}}{-\frac{1}{2}}\right|$ (2)

$=7$

$\theta=81°52'$

b) $u=5-x^2$

$\frac{du}{dx}=-2x$ (2)

$du=-2x\,dx$

$\int\frac{x}{(u)^3}\cdot\frac{du}{-2x}$

$=-\frac{1}{2}\int u^{-3}$

$=-\frac{1}{4}u^{-2}$

$=-\frac{1}{4(5-x^2)^2}+C$

(2)

c) $u=x-3$

$du=dx$

limits

$x=4$ $\quad x=3$

$u=1$ $\quad u=0$

$\int_0^1 (u+3)u^{1/2}\,du$

$\int_0^1 u^{3/2}+3u^{1/2}\,du$

$\left[\frac{2}{5}u^{5/2}+2u^{3/2}\right]_0^1$ (3)

$\left[\frac{2}{5}+2-0\right]=2\frac{2}{5}$

d) $x\neq 0$ as critical pt

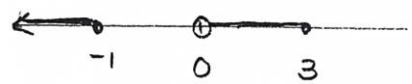
solve the equation

$\frac{2x+3}{x}=x$ (3)

$2x+3=x^2$

$x^2-2x-3=0$

$(x-3)(x+1)=0$



test $x=1$ $\quad$ test $x=-2$

$\frac{5}{1}\geq 1$ ✓ $\quad \frac{-4+3}{-2}\geq -2$ ✓

$x\leq -1$ or $0<x\leq 3$

e) $\int_{\frac{\pi}{2}}^{\pi} \frac{1-\cos 2x}{2} + x \, dx$

$\left[\frac{1}{2}x - \frac{\sin 2x}{4} + \frac{x^2}{2}\right]_{\frac{\pi}{2}}^{\pi}$

$= \left[\frac{\pi}{2} + \frac{\pi^2}{2}\right] - \left[\frac{\pi}{4} + \frac{\pi^2}{8}\right]$

$= \frac{3\pi^2 + 2\pi}{8}$ (3)

f) $y^2 + x^2 - 2y - 24 = 0$

$(5\cos\theta+1)^2 + (5\sin\theta)^2 - 2(5\cos\theta+1) - 24$

$25\cos^2\theta + 10\cos\theta + 1 + 25\sin^2\theta$

$-10\cos\theta - 2 - 24$

$25(\cos^2\theta + \sin^2\theta) - 2 - 24 + 1$

$25 - 2 - 24 + 1$

$= 0$ RHS

QUESTION 9.

a) i) $M = \frac{2p+2q}{2}, \frac{p^2+q^2}{2}$

$= \left(p+q, \frac{p^2+q^2}{2}\right)$ (1)

ii) Find gradiets OP, OQ

$m_{OP} = \frac{p^2-0}{2p}$ $\quad m_{OQ} = \frac{q^2-0}{2q}$

since rt < $m_{OP} \times m_{OQ} = -1$

$\frac{p}{2} \times \frac{q}{2} = -1$ (2)

$\therefore pq = -4$. (3)

iii) for M consider (x, y)

$x = p+q \quad y = \frac{p^2+q^2}{2}$

$\therefore y = \frac{(p+q)^2 - 2pq}{2}$

$= \frac{x^2 - 2pq}{2}$

since $pq = -4$

$y = \frac{x^2+8}{2}$ (2)

9 b) M is where it cuts y axis

$x = 0$

$py - 2p - p^3 = 0$

$y = p^2 + 2$

$\therefore M(0, 2+p^2)$

2) $R = (x, y)$

we know M is midpoint PR

$\therefore 0 = \frac{x+2p}{2} \qquad 2+p^2 = \frac{y+p^2}{2}$

$0 = x+2p \qquad 4+2p^2 = y+p^2$

$\therefore -2p = x \qquad 4+p^2 = y$

$p = \frac{-x}{2}$

for Locus $y = 4+p^2$

$y = 4 + \left(-\frac{x}{2}\right)^2$

$y = 4 + \frac{x^2}{4}$

$4y = 16 + x^2$

$\therefore x^2 = 4y - 16$

$x^2 = 4(y-4)$

vertex $(0, 4)$

c) $\cos x - x = 0$ (4)

$x = 0.7$

i) $\cos 0.7 - 0.7 = 0.06 > 0$

$x = 0.8$

$\cos 0.8 - 0.8 = -0.103 < 0$

sign change $\therefore$ a root exists between $x = 0.7$ and 0.8

ii) $x = 0.75$

$f(x) = \cos x - x$

$= \cos 0.75 - 0.75$

$= -0.0183$

$f'(x) = -\sin x - 1$

$= -\sin 0.75 - 1$

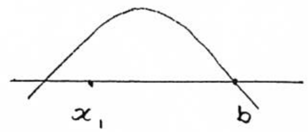
$= -1.68$

$\therefore x_1 = 0.75 - \frac{-0.0183}{-1.68}$

$= 0.75 - .0108$

$= 0.739$

iii)



- choosing an approximat^n when a stationary point exists between it and the actual root at $x = b$
- choosing x_1 at a stationary point.
- because of the slope and curvature x_1 is not close enough to the root making x_2 worse than x_1