HORNSBY GIRLS' HIGH SCHOOL

MATHEMATICS

HSC ASSESSMENT TASK 2

TERM 1 2001

Extension 1

Time allowed : 70 minutes
(Plus 2 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt all questions.
- All questions are of equal value.
- All necessary working should be shown. Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- Each question is to be handed in separately, clearly marked Question 1, Question 2, etc...
- Write your name on every page.

These outcomes are being assessed in this task :

HE1 – appreciates interrelationships between ideas drawn from different areas of mathematics
HE3 – uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion, or exponential growth and decay
HE4 –uses the relationship between functions,inverse functions and their derivatives
HE5 – applies the chain rule tp problems including those involving velocity and acceleration as functions of displacement
HE6 – determines integrals by reduction to a standard form through a given substitution
HE7 – evaluates mathematical solutions to problems and communicates them in an appropriate form

QUESTION 1 - (20 Marks) *START A NEW PAGE*

a) Solve the equation $\sin^2 x = \sin x$ for $-2\pi \le x \le 2\pi$

b) Show that $\tan^{-1}\left(\frac{3}{4}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$

c) Show that $\dfrac{e^{2x}}{e^{2x}+1} = \dfrac{e^x}{e^x + e^{-x}}$

Use this result to calculate the area enclosed between the curve $y = \dfrac{e^x}{e^x + e^{-x}}$, the x-axis and the ordinates $x = 0$ and $x = 3$.

d) A cone has a volume given by $v = \dfrac{3\pi h^2}{10}$, where h is the height of liquid in the cone. If the height of liquid is increasing at a rate of 4cms^{-1}, find the rate of increase in volume when the height of liquid is 15cm.

e) A particle moves with a constant acceleration of 9m/s^2. Given that the velocity is 12m/s when the particle is 6metres from the origin, find:

(i) an expression for velocity in terms of displacement
(ii) the velocity when $x = 0$.

f) (i) Find the greatest domain over which $f(x) = (x+1)^2 - 2$ has an inverse function
(ii) Find the inverse function
(iii) What is the domain of f^{-1}.

QUESTION 2 - (20 Marks) *START A NEW PAGE*

a) Given that $y = \log_e(\cos x)$, find $\dfrac{dy}{dx}$

Hence find $\displaystyle\int_0^{\frac{\pi}{3}} \tan x \; dx$

b) Differentiate with respect to x

(i) $y = 3\sin^{-1} x$

(ii) $y = x^2 \cos^{-1}(1-x)$

(c) Newton's law of cooling states that for an object placed in surroundings at constant temperature, the rate of change of the temperature of the object is proportional to the difference between the temperature of the object and the surroundings, i.e.

$$\frac{dT}{dt} = k(T - T_0)$$

where T_0 is the temperature of the surroundings, T is the temperature of the object at any time, t, and k is a constant.

(i) Show that $T = T_0 + A e^{kt}$ is a solution of this equation

(ii) A ball bearing initially at 100^0C is dropped into a large vat of oil. After 30 seconds, the temperature of the ball bearing has dropped to 80^0C. If the temperature of the oil is constant at 30^0C, find the value of A and k.

(iii) Calculate how long it would take for the ball bearing to reach 35^0C.

d) A particle is moving in SHM and its acceleration is given by $\ddot{x} = -x$

(i) Show that $x = 7\sin(t + \pi)$ is a possible equation for the displacement of the particle

(ii) Write down the amplitude and period of the motion.

(iii) Where is the particle when the velocity is 7cms^{-1}?

QUESTION 3 - (20 Marks) *START A NEW PAGE*

a) Find $\displaystyle\int_0^{\frac{\pi}{2}} \cos\theta \sin^3\theta \; d\theta$

b) Integrate the following

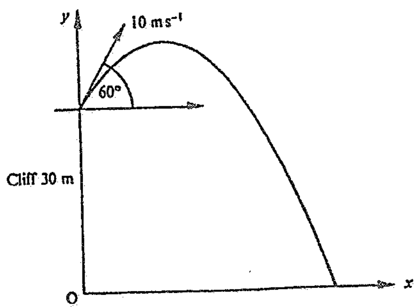
(i) $\displaystyle\int \frac{dx}{\sqrt{7-x^2}}$

(ii) $\displaystyle\int \frac{dx}{x^2 + 6x + 25}$

c) Evaluate

$$\int_0^{3/4} \frac{dx}{\sqrt{3 - 4x^2}}$$

d) A stone is thrown into the sea from the top of a 30m vertical cliff with velocity 10ms^{-1} at an angle of elevation of $60°$.



(Figure not to scale)

(i) Taking the origin as point O, derive the equations (in exact form) for the horizontal (x) and vertical (y) components of the stone's displacement from O after t seconds (neglect air resistance and assume acceleration due to gravity is 10ms^{-2}).

(ii) Calculate the maximum height of the stone and how far it lands from the foot of the cliff.

e) A particle moves in a straight line so that its velocity v at a position x is given by: $v^2 = 4(3 + 2x - x^2)$

(i) Show that $\ddot{x} = -4(x-1)$ and hence is simple harmonic motion.
(ii) State the centre of motion
(iii) What is the amplitude of the motion?
(iv) What is the period of the motion?
(iv) What is the maximum speed of the particle?

3 UNIT 2001 Term 1

1
(a) $\sin^2 x - \sin x = 0$ for $-2\pi \le x \le 2\pi$

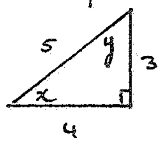
$\sin x(\sin x - 1) = 0$

$\sin x = 0$ $\quad$ $\sin x = 1$

$x = -2\pi, -\pi, 0, \pi, 2\pi, \frac{\pi}{2}, -\frac{3\pi}{2}$

(b) To show $\tan^{-1}\frac{3}{4} + \cos^{-1}\frac{3}{5} = \frac{\pi}{2}$

Let $x = \tan^{-1}\frac{3}{4}$, $\tan x = \frac{3}{4}$ | Let $y = \cos^{-1}\frac{3}{5}$, $\cos y = \frac{3}{5}$



∴ $x + y = \frac{\pi}{2}$

∴ $\tan^{-1}\frac{3}{4} + \cos^{-1}\frac{3}{5} = \frac{\pi}{2}$

(c) LHS $= \frac{e^{2x}}{e^{2x}+1}$

Divide all by e^x

LHS $= \frac{e^x}{e^x + e^{-x}}$

= RHS

$A = \int_0^3 \frac{e^x}{e^x + e^{-x}}\,dx$

$= \int_0^3 \frac{e^{2x}}{(e^{2x}+1)^1}\,dx$

$= \frac{1}{2}\left[\ln(e^{2x}+1)\right]_0^3$

$= \frac{1}{2}\left[\ln(e^6+1) - \ln(2)\right]$

$= \frac{1}{2}\ln\frac{e^6+1}{2}$

(d) $V = \frac{3\pi h^2}{10}$ and $\frac{dh}{dt} = 4$.

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$= \frac{3\pi h}{5} \times 4$

$= 36\pi$ cm³/sec

(e). $\ddot{x} = 9$.

Given $v = 12$, $x = 6$

Formula $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

∴ $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 9$

$\frac{1}{2}v^2 = 9x + c$

$x=6$, $v=12$: $72 = 54 + c$

$c = 18$

∴ $\frac{1}{2}v^2 = 9x + 18$

$v^2 = 18x + 36$

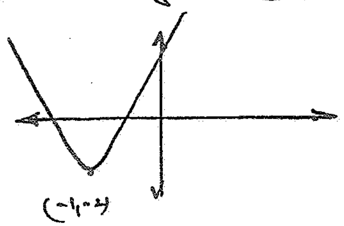
$v = \sqrt{18x+36}$

NB only $+\sqrt{\ }$ since $v=12$, $x=6$

when $x = 0$

$v = 6$ m/sec.

(f). Sketch $f(x) = (x+1)^2 - 2$



Greatest domain for this to be a 1-1 function [ie for an inverse to exist]

either $x \ge -1$

OR $x \le -1$

for $x \ge -2$.

Let inverse: $y = (x+1)^2 - 2$.

$x = (y+1)^2 - 2$.

$(y+1)^2 = x + 2$

$y + 1 = \sqrt{x+2}$

Inverse Funct. $y = \sqrt{x+2} - 1$

(ii) ie $f^{-1}(x) = \sqrt{x+2} - 1$

(iii). Domain of $f^{-1}(x)$ is the same as the Range of $f(x)$

(ie) (ie) $x \ge -2$.

Question 2.

(a) $y = \ln(\cos x)$

$\frac{dy}{dx} = \frac{1}{\cos x} \times -\sin x$

$= -\frac{\sin x}{\cos x}$

$= -\tan x$.

∴ $\int_0^{\frac{\pi}{3}} \tan x\,dx$

$= \int_0^{\frac{\pi}{3}} -\frac{\sin x}{\cos x}\,dx$

$= -\left[\ln\cos x\right]_0^{\frac{\pi}{3}}$

$= -\left(\left[\ln\frac{1}{2}\right] - \left[\ln 1\right]\right)$

$= -\ln\frac{1}{2}$

$= -\ln 2^{-1}$

$\ln 2$.

(b). (i) $y = 3\sin^{-1}x$

$\frac{dy}{dx} = 3 \times \frac{1}{\sqrt{1-x^2}}$ $\quad -1 < x < 1$

$= \frac{3}{\sqrt{1-x^2}}$ for $-1 <$

(ii) $y = x^2\cos^{-1}(1-x)$

Use Product formula.

$\frac{dy}{dx} = \frac{x^2 \times -1 \times -1}{\sqrt{1-(1-x)^2}} + 2x$

$= \frac{x^2}{\sqrt{x(2-x)}} + 2x\cos^{-1}($

NB Domain $x(2-x) > 0$.

for $0 < x < 2$

(c).
(i) $T = T_0 + Ae^{kt}$.

$\frac{dT}{dt} = kAe^{kt}$

$= k(T - T_0)$

(ii) $T = 30 + Ae^{kt}$.

Sub $t = 0$, $T = 100$: $100 = 30 + Ae^{k0}$.

$70 = A$

∴ $T = 30 + 70e^{kt}$.

Sub $T = 80$, $t = 30$: $80 = 30 + 70e^{30k}$

$\frac{5}{7} = e^{30k}$

$k = \frac{1}{30}\ln\frac{5}{7}$

$35 = 30 + 70e^{\frac{1}{10}\ln\frac{5}{7}\times t}$

$5 = 70e^{\frac{1}{10}\ln\frac{5}{7}\times t}$

$t = 3$ mins 55 secs ✓

(i) $x = 7\sin(t+\pi)$

$\dot{x} = 7\cos(t+\pi)$

$\ddot{x} = -7\sin(t+\pi)$

$\ddot{x} = -x.$ ✓

It is (SHM) as it satisfies the definition $\ddot{x} = -n^2x$ when $n=1$
(SHM) about the origin

(ii) Amplitude $= 7$ ✓

Period $= \frac{2\pi}{n}$ when $n=1$

Period $= 2\pi$ seconds ✓

(iii) $v^2 = n^2(A^2-x^2)$ about the origin

$49 = 1(49-x^2)$

$x=0$ when $v=7$ ✓

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estion 3.

3 Unit Assessment No 2 2001

Question 3 (20 marks)

(a) $\int_0^{\pi/2} \cos\theta \sin^3\theta \, d\theta$

$= \frac{1}{4}\times\left[\sin^4\theta\right]_0^{\pi/2}$

$= \frac{1}{4}\times\left[\sin^4\frac{\pi}{2}\right] - \left[\sin^4 0\right]$

$= \frac{1}{4}$ ✓ (2)

(b) (i) $\int \frac{dx}{\sqrt{\sqrt{7}^2 - x^2}}$

$= \sin^{-1}\frac{x}{\sqrt{7}} + c$ ✓ (3)

(ii) $\int \frac{dx}{x^2+6x+25}$

$= \int \frac{dx}{(x+3)^2+16}$

$= \int \frac{dx}{4^2+(x+3)^2}$

$= \frac{1}{4}\tan^{-1}\frac{x+3}{4} + c$ ✓ (3)

(c) $\int_0^{3/4} \frac{dx}{\sqrt{3-4x^2}}$

$= \frac{1}{2}\int_0^{3/4} \frac{dx}{\sqrt{(\frac{\sqrt{3}}{2})^2 - x^2}}$

$= \frac{1}{2}\left[\sin^{-1}\frac{x}{\frac{\sqrt{3}}{2}}\right]_0^{3/4}$

$= \frac{1}{2}\left[\sin^{-1}\frac{2x}{\sqrt{3}}\right]_0^{3/4}$

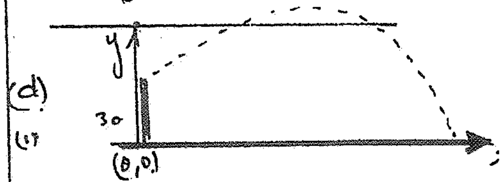
$= \frac{1}{2}\times\left[\sin^{-1}\frac{3}{2\sqrt{3}}\right] - \left[\sin^{-1}0\right]$

$= \frac{1}{2}\left\{\sin^{-1}\frac{\sqrt{3}}{2}\right\}$

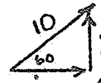
$= \frac{1}{2}\times\frac{\pi}{3}$

$= \frac{\pi}{6}$ ✓ (3)

(d) (i)



When t=0



$\dot{y} = 10\sin 60 = 5\sqrt{3}$

$\dot{x} = 10\cos 60 = 5$

Vertical Motion	Horizontal Motion
$\ddot{y} = -10.$	$\ddot{x} = 0$
$\dot{y} = -10t + 5\sqrt{3}$ (1)	$\dot{x} = 5$ (1)
$y = -5t^2 + 5\sqrt{3}t + 30$ ✓	$x = 5t.$ ✓

(ii) Upward Time $\dot{y} = 0$ $t = \frac{\sqrt{3}}{2}$

$\therefore$ $y = -5\times\frac{3}{4} + 5\sqrt{3}\times\frac{\sqrt{3}}{2} +$

$y = -\frac{15}{4} + \frac{15}{2} + 30$

(1) $y = 33.75$m $(33\frac{3}{4})$

(iii) To find Total Time Put $y = 0$

$-5t^2 + 5\sqrt{3}t + 30 = 0$

$t^2 - \sqrt{3}t - 6 = 0$

$t = \frac{\sqrt{3} \pm \sqrt{3+24}}{2}$

$t = 2\sqrt{3}$

$x = 5\times 2\sqrt{3}$ (1)

$x = 10\sqrt{3}$ (17.32 m). ✓

(e)

$v^2 = 4(3+2x-x^2)$

(i)

(i) $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$

$= \frac{d}{dx}\, 2(3+2x-x^2)$

$= 2(2-2x) = 4-4x$

$= -4(x-1)$ ✓ (1)

$\ddot{x} = -n^2(x-h)$

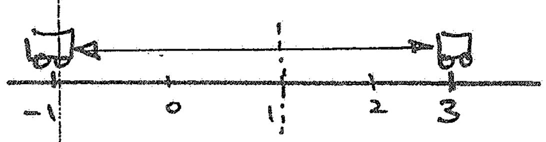
(ii) SHM Centre is $x=1$ (1)

(iii) Sub $v=0$ $x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3 \quad x=-1$

Motion stops at $x=3$ and $x=-1$



Amplitude = 2 units ✓ (1)

(iv) Period $= \frac{2\pi}{n}$ $n=2$.

$= \pi$ Seconds ✓ (1)

(v) Max speed is at Centre of Motion

(ie) when $x=1$

$v^2 = 4(3+2-1)$

$v^2 = 4(4)$

$v = \pm 4$

Max speed = 4 m/sec ✓ (1)