HURLSTONE AGRICULTURAL HIGH SCHOOL

PRO PATRIA

MATHEMATICS EXTENSION 1

2013

HSC

Assessment Task 2

Examiners: P. Biczo, J. Dillon, S. Faulds, S. Gutesa, G. Huxley, B. Morrison

General Instructions

- Reading time – 5 minutes.
- Working time – 90 minutes.
- Attempt **all** questions.
- Board approved calculators and Math Aids may be used.
- This examination must **NOT** be removed from the examination room

- **Section A** consists of five (5) multiple choice questions worth 1 mark each. Fill in your answer on the multiple choice answer sheet provided.
- **Section B** requires all necessary working to be shown in every question. This section consists of five (5) questions worth 9 marks each. Marks may not be awarded for careless or badly arranged work.
  Each question is to be started in a new answer booklet. Additional booklets are available if required.

Name : 24397548

Class : 12A3

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SECTION A – 5 multiple choice questions (1 mark each)

Answer on the sheet provided. This sheet may be torn off the back.

Question 1

Given α, β and γ are the roots of the equation $2x^3+5x^2-x-3=0$, then the value of $\alpha\beta+\alpha\gamma+\beta\gamma$ is

A. $-\frac{5}{2}$ **B.** $-\frac{1}{2}$ **C.** $\frac{1}{2}$ **D.** $-\frac{3}{2}$

Question 2

The area between the $x-$axis and the curve $y=x(x-2)(x-3)$ bounded by the straight lines $x=0$ and $x=3$ is best determined by:

A. $\int_0^3 x(x-2)(x-3)dx$

B. $\int_0^2 x(x-2)(x-3)dx+\int_2^3 x(x-2)(x-3)dx$

C. $\int_0^2 x(x-2)(x-3)dx+\left|\int_2^3 x(x-2)(x-3)dx\right|$

D. $\left|\int_0^3 x(x-2)(x-3)dx\right|$

Question 3

How many distinct permutations of the letters of the word 'ATTAINS' are possible in a straight line when the word begins and ends with the letter T?

A. 60 **B.** 120 **C.** 360 **D.** 1260

Question 4

What is the exact value of $\tan 75^\circ$?

A. $2-\sqrt{3}$ **B.** $4-\sqrt{3}$ **C.** $2+\sqrt{3}$ **D.** $4+\sqrt{3}$

Question 5

A and ***B*** are the points (–3, 2) and (5, 7) respectively. The point ***P*** $(-1, 3\frac{1}{4})$ divides the interval ***AB***:

A. Internally in the ratio 1:3

B. Externally in the ratio 2:1

C Externally in the ratio 1:2

D. Internally in the ratio 3:1

SECTION B

Question 6 (9 marks) **Use a SEPARATE writing booklet** — **Marks**

(a) Prove by mathematical induction: **3**

$$\sum_{r=1}^{n} r.r! = (n+1)! - 1$$

(b) **(i)** Find the x-value of the point where the curves $y = 5 - x^2$ and $y = (x-1)^2$ intersect in the first quadrant. **1**

(ii) Hence, find the acute angle between the two curves at their point of intersection in the first quadrant to the nearest degree **2**

(c) Solve the inequality: **3**

$$\frac{x^2 - 5x}{x - 4} \leq 3$$

Question 7 (9 marks) **<u>Use a SEPARATE writing booklet</u>** **Marks**

(a) Show that $\dfrac{1+\cos 2\theta}{\sin 2\theta}=\cot\theta$ **2**

(b) Mr Gee has five different textbooks on his desk:
One for Extension 2, one for Extension 1, one for Mathematics, one for Year 8 and one for Year 7

During breaks between lessons he stacks two, three, four or five textbooks on top of one another to form a vertical tower.

(i) How many different towers can be formed, that are three textbooks high? **1**

(ii) How many different towers can he form in total? **2**

(c) Five players are selected at random from three sporting teams *A*, *B* and *C*.
Each team consists of seven players numbered 1 to 7.

(i) Two brothers play for different teams. What is the probability that of the five selected players, both brothers are selected? **2**

(ii) Jason wanted to find the number of ways to select five players so that no team misses out in the selection.

His answer was

$$^3C_1\,{}^7C_1\,{}^7C_1\,{}^7C_3+{}^3C_1\,{}^7C_1\,{}^7C_2\,{}^7C_2$$

Explain why this expression is correct. **2**

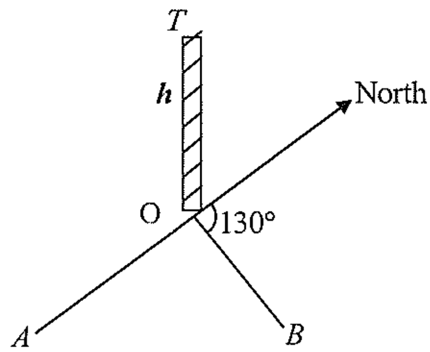
Question 8 (9 marks) <u>**Use a SEPARATE writing booklet**</u> **Marks**

(a) On 1st January 2013 James commenced a savings account where he deposited \$1200 per month and earned 6% p.a. compounded monthly. His aim was to have \$100 000 saved by the end of 2020.

(i) Show that by 1st April 2018 James will have \$89 047.24 **2**

(ii) James realizes he no longer needs to deposit \$1200 to achieve his goal. What is the minimum amount that James must continue to deposit into his savings in order to achieve his goal by the end of 2020? **2**

(b) Mrs Hackett wanted to measure the height (*h*) of a tower. Using her new ultra sonic distance measurer given to her by her brother for her birthday, she measured the angle of elevation to the tower from two different points.
At a point A due south of the tower, the angle of elevation to T, is $25°$.
She then walked 100 m to point B, on a bearing of $130°$ from the tower, where the angle of elevation is $30°$.



(i) Copy or trace the diagram into your writing booklet showing all information on your diagram. **1**

(ii) Show that $OA = h\tan 65°$ **1**

(iii) Show that $h^2 = \dfrac{10000}{\tan^2 65° + \tan^2 60° - 2\cos 50° \tan 65° \tan 60°}$ **3**

Question 9 (9 marks) **Use a SEPARATE writing booklet** **Marks**

(a) Show that the polynomial $P(x)=x^4-2x^3+7x-6$ is divisible by $(x-1)$. **1**

(b) A root of the equation $e^x-x^2=0$ lies near $x=-0.5$. Taking $x=-0.5$ as a first approximation to the root, use Newton's Method to obtain a second approximation, correct to two decimal places. **2**

(c) The equation of the chord PQ of the parabola $x^2=4y$, where P is the point $(2p,p^2)$ and Q the point $(2q,q^2)$ is $y=\frac{(p+q)}{2}x-pq$. (YOU DO NOT NEED TO SHOW THIS)

(i) Find the co-ordinates of the midpoint M of PQ. **1**

(ii) If the chord passes through the point (0, 2), find the locus of M. **2**

(d) (i) Find $\frac{d(xe^{2x})}{dx}$ **1**

(ii) Hence, evaluate $\int_0^1 2xe^{2x}dx$ **2**

Question 10 (9 marks) **Use a SEPARATE writing booklet** **Marks**

(a) Find the area between the curve **2**

$y=\sqrt{x+1}$, and the $x-$axis between $x=-1$ and $x=3$

(b) The area between the semicircle

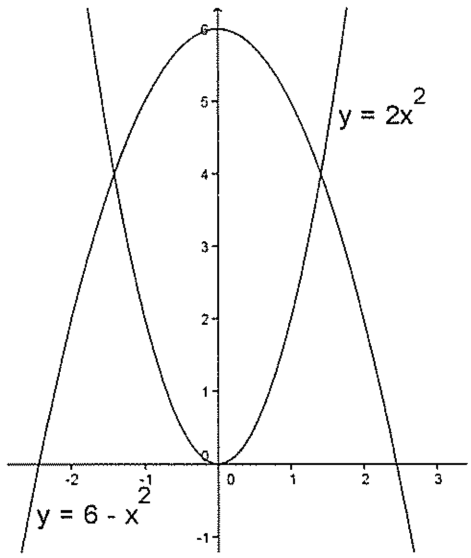
$y=\sqrt{9-x^2}$ and the $x-axis$ is to be approximated using Simpson's rule with the 5 intervals determined by: $x=-3$, $x=-1\cdot5$, $x=0$, $x=1\cdot5$ and $x=3$.

(i) Find the approximate area using the five intervals to 3 significant figures. **2**

(ii) Using the approximate area determine an approximate value of π **1**

(iii) What is the percentage error in this value of π (to 1 decimal place) **1**

(c)



The area between the curves $y=2x^2$ and $y=6-x^2$ is rotated about the y-axis

(i) Find the x values of the points of intersection of the two curves **1**

(ii) Find the volume of the solid of revolution. **2**

End of Examination

HURLSTONE AGRICULTURAL HIGH SCHOOL

MATHEMATICS EXTENSION 1

2013

HSC

Assessment Task 2

Marking Scheme:

Extension 1 Task 2 HSC 2013 Multiple Choice Answers

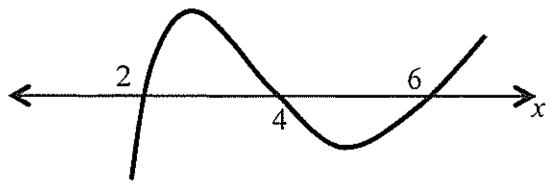
Question 1:	B
Question 2:	C
Question 3:	A
Question 4:	C
Question 5:	A

Year 12 Mathematics Extension 1 Half Yearly Examination 2013	
Question No. 6	Solutions and Marking Guidelines

Outcomes Addressed in this Question

HE2 uses inductive reasoning in the construction of proofs
PE3 solves problems involving permutations and combinations, **inequalities**, polynomials, circle geometry and parametric representations
PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

Outcome	Solutions	Marking Guidelines
HE2	**(a)** 1. Prove true for $n = 1$ $\text{L.H.S} = \sum_{r=1}^{1} r.r! = 1.1! = 1$ $\text{R.H.S} = (1+1)! - 1 = 1 = \text{L.H.S}$ $\therefore$ True for $n = 1$ 2. Assume true for $n = k$. ie. Assume $\sum_{r=1}^{k} r.r! = (k+1)! - 1$ Prove true for $n = k+1$ ie. Prove $\sum_{r=1}^{k+1} r.r! = (k+2)! - 1$ $\text{L.H.S} = \sum_{r=1}^{k+1} r.r!$ $= \sum_{r=1}^{k} r.r! + (k+1)(k+1)!$ $= (k+1)! - 1 + (k+1)(k+1)!$ $= (k+1)![1+(k+1)] - 1$ $= (k+1)!(k+2) - 1$ $= (k+2)! - 1$ $= \text{R.H.S}$ 3. Hence, by the priciple of mathemtical induction, the result is true for all values of $n \geq 1$.	**3 marks** Correct solution showing full reasoning. **2 marks** Substantial progress towards full solution including proof for $n = 1$. **1 mark** Proves result is true for $n = 1$.
PE6	**(b) (i)** Point of intersection occurs when: $5 - x^2 = (x-1)^2$ $= x^2 - 2x + 1$ $0 = 2x^2 - 2x - 4$ $= x^2 - x - 2$ $= (x+1)(x-2)$ $\therefore$ Point in first quadrant: $(2,1)$	**1 mark** Correct solution giving point in first quadrant.
PE6	**(ii)** $y = 5 - x^2$, $\frac{dy}{dx} = -2x$ $y = (x-1)^2$, $\frac{dy}{dx} = 2x - 2$ Gradients of curves when $x = 2$ $m_1 = -2 \times 2 = -4$ $m_2 = 2 \times 2 - 2 = 2$ Acute angle (θ) between curves when $x = 2$ $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$ $= \frac{-4-2}{1+(-4\times 2)}$ $= \frac{-6}{-7}$ $\therefore \theta = \tan^{-1}\left(\frac{6}{7}\right)$ $= 41°$ (to nearest degree)	**2 marks** Correct solution. **1 mark** Substantial progress towards correct solution including gradients of both curves.

PE3	**(c) (i)** $\dfrac{x^2-5x}{x-4} \le 3$ Multiplying both sides of the inequality by $(x-4)^2$ $(x^2-5x)(x-4) \le 3(x-4)$ $(x^2-5x)(x-4)-3(x-4) \le 0$ $(x-4)\left[(x^2-5x)-3(x-4)\right] \le 0$ $(x-4)\left[x^2-5x-3x+12\right] \le 0$ $(x-4)\left[x^2-8x+12\right] \le 0$ $(x-4)(x-6)(x-2) \le 0$ Sketch of $y = (x-4)(x-6)(x-2)$  From graph: $x \le 2$, $4 < x \le 6$ (Note: $x \ne 4$)	**3 marks** Correct solution. **2 marks** Substantial progress towards correct solution. **1 mark** Recognises that both sides of the inequality must be multiplied by a positive, **OR** that the inequality must be considered in cases. Either approach must include a correct start to the solution.

Year 12	Mathematics Extension 1	Half Yearly Examination 2013
Question 7	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

P2 provides reasoning to support conclusions which are appropriate to the context
PE2 uses multi-step deductive reasoning in a variety of contexts
PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations

Part	Solutions	Marking Guidelines
(a) P3,PE2	$\text{LHS} = \dfrac{1+\cos 2\theta}{\sin 2\theta}$ $= \dfrac{1+2\cos^2\theta - 1}{2\sin\theta\cos\theta}$ $= \dfrac{2\cos^2\theta}{2\sin\theta\cos\theta}$ $= \dfrac{\cos\theta}{\sin\theta}$ $= \cot\theta$ $= \text{RHS}$	**Award 2** ~ Correct solution. **Award 1** ~ Substantial progress towards solution
(b) (i) PE3	Number of towers $= 5\times 4\times 3 = 60$	**Award 1** ~ Correct solution
(ii) PE3	Number of towers 2 textbooks high $= 5\times 4 = 20$ 3 textbooks high $= 5\times 4\times 3 = 60$ 4 textbooks high $= 5\times 4\times 3\times 2 = 120$ 5 textbooks high $= 5\times 4\times 3\times 2\times 1 = 120$ $\therefore$ Number of towers that can be made $= 320$	**Award 2** ~ Correct solution. **Award 1** ~ Substantial progress towards solution
(c) (i) PE3	Place the two brothers, then choose 3 from 19 Number of ways $= {}^1C_1 \times {}^1C_1 \times {}^{19}C_3$ Number of unrestricted ways $= {}^{21}C_5$ $\therefore$ Probability $= \dfrac{{}^1C_1\times {}^1C_1 \times {}^{19}C_3}{{}^{21}C_5} = \dfrac{1}{21}$	**Award 2** ~ Correct solution. **Award 1** ~ Substantial progress towards solution
(ii) PE3	There are two patterns: 1, 1, 3 or 1, 2, 2 **1, 1, 3** one player from each of two teams and three from the other ${}^7C_1 \times {}^7C_1 \times {}^7C_3$ But the team with three player can be selected in ${}^3C_1$ ways $\therefore$ Number of ways $= {}^3C_1 \times {}^7C_1 \times {}^7C_1 \times {}^7C_3$	**Award 2** ~ Correct solution. **Award 1** ~ Substantial progress towards solution

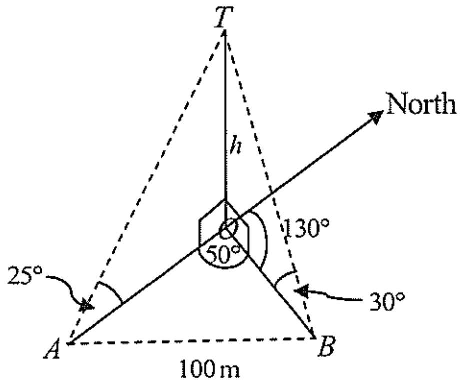
	1, 2, 2 one player from one team and two players from the other two teams $^7C_1 \times {}^7C_2 \times {}^7C_2$ But the team with one player can be selected in $^3C_1$ ways $\therefore$ Number of ways = $^3C_1 \times {}^7C_1 \times {}^7C_2 \times {}^7C_2$ $\therefore$ Total Number of ways $= {}^3C_1 \times {}^7C_1 \times {}^7C_1 \times {}^7C_3 + {}^3C_1 \times {}^7C_1 \times {}^7C_2 \times {}^7C_2$	

Year 12 Task 2 Question No. 8	Mathematics-Extension 1 Solutions and Marking Guidelines	Examination 2013

Outcomes Addressed in this Question

H5 - applies appropriate techniques from the study of calculus, geometry, probability, *trigonometry* and *series* to solve problems

Outcome	Solutions	Marking Guidelines
H5	**a)** *Similar to a superannuation question **i)** $P=\$1200,\ r=0.005,\ n=63$ $A_n=P(1+r)^n$ $A_1=1200(1.005)^{63}$ $A_2=1200(1.005)^{62}$ $A_3=1200(1.005)^{61}$ ... $A_{63}=1200(1.005)$ By 1st April 2018 James will have: $A_1+A_2+A_3+...+A_{63}$ $=1200(\underbrace{1.005+1.005^2+1.005^3+...+1.005^{63}}_{\text{G.P. with } a=1.005,\ r=1.005,\ n=63,\ S_n=\frac{a(r^n-1)}{r-1}})$ $=1200\left(\frac{1.005(1.005^{63}-1)}{1.005-1}\right)$ $=\$89\,047.24$	**(2 marks)** correct solution **(1 mark)** substantial progress towards correct solution
	ii) From 1 April 2018 to end of 2020, $n=33$ $89\,047.24\times1.005^{33}=\$104978.53$ Therefore, without making any further deposits, James would earn enough interest on the amount he has already saved, to take his total savings over \$100 000 by the end of 2020.	**(2 marks)** correct solution **(1 mark)** substantial progress towards correct solution
	Or Still required to save $=\$100000-\$89047.24$ $=\$10952.76$ (by the end of 2020) $A_1=R(1.005)^{33}$ $A_2=R(1.005)^{32}$ $A_3=R(1.005)^{31}$ ... $A_{33}=R(1.005)$ $10952.76=A_1+A_2+A_3+...+A_{33}$ $10952.76=R(\underbrace{1.005+1.005^2+1.005^3+...+1.005^{33}}_{\text{G.P. with } a=1.005,\ r=1.005,\ n=33,\ S_n=\frac{a(r^n-1)}{r-1}})$ $10952.76=R\left(\frac{1.005(1.005^{33}-1)}{1.005-1}\right)$ $R=\$304.58$ $\therefore$ James needs to deposit at least \$304.58 to achieve his goal by the end of 2020.	**This solution is not entirely correct as it does not consider the interest earned on the \$89 047.24 that has already been saved.

H5		
	b) **(i)**  	**(1 mark)** Diagram showing all information.
	(ii) In ΔTOA, $\tan 25° = \dfrac{h}{OA}$ $OA = \dfrac{h}{\tan 25°}$ $\therefore OA = h\tan 65°$	**(1 mark)** Correct solution
	(iii) Similarly to (ii), In ΔTOB, $OB = h\tan 60°$ By the cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$ $100^2 = \left(h\tan 65°\right)^2 + \left(h\tan 60°\right)^2 - 2\left(h\tan 65°\right)\left(h\tan 60°\right)\cos 50°$ $10000 = h^2\tan^2 65° + h^2\tan^2 60° - 2h^2\tan 65°\tan 60°\cos 50°$ $10000 = h^2\left(\tan^2 65° + \tan^2 60° - 2\tan 65°\tan 60°\cos 50°\right)$ $\dfrac{10000}{\left(\tan^2 65° + \tan^2 60° - 2\tan 65°\tan 60°\cos 50°\right)} = h^2$ $\therefore h^2 = \dfrac{10000}{\left(\tan^2 65° + \tan^2 60° - 2\cos 50°\tan 65°\tan 60°\right)}$	**(3 marks)** correct solution **(2 marks)** substantial progress towards correct solution **(1 mark)** Some progress towards correct solution

Year 12 Half Yearly Question No.9	Extension 1 Mathematics Solutions and Marking Guidelines	Examination 2013

Outcomes Addressed in this Question

PE3 solves problems involving polynomials and parametric representations.
PE5 determines derivatives which require the application of more than one rule of differentiation.
H5 applies appropriate techniques from the study of calculus to solve problems

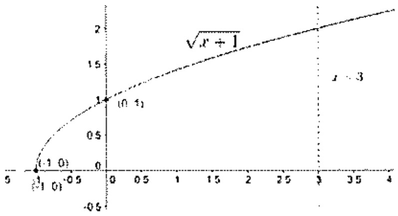
Outcome	Solutions	Marking Guidelines
PE3	a) $P(x)=x^4-2x^3+7x-6$ $P(1)=1-2+7-6=0.$ Since $P(1)=0,\ P(x)$ is divisible by $x-1$	1 mark : correct solution
H5	b) Let $f(x)=e^x-x^2.$ $f(-0.5)=e^{-0.5}-(-0.5)^2=e^{-0.5}-0.25.$ $f'(x)=e^x-2x.$ $f'(-0.5)=e^{-0.5}-2\times-0.5=e^{-0.5}+1.$ A better approximation is given by $-0.5-\dfrac{f(-0.5)}{f'(-0.5)}$ $=-0.5-\dfrac{e^{-0.5}-0.25}{e^{-0.5}+1}$ $=-0.72$	2 marks: correct solution 1 mark : significant progress towards correct solution
PE3	c) (i) Midpoint $M=\left(\dfrac{2p+2q}{2},\dfrac{p^2+q^2}{2}\right)=\left(p+q,\dfrac{p^2+q^2}{2}\right)$	1 mark : correct answer
	(ii) (0, 2) lies on $y=\dfrac{(p+q)}{2}x-pq.$ $\therefore 2=0-pq.$ $\therefore pq=-2.$ At M, $\begin{cases}x=p+q\\ y=\dfrac{p^2+q^2}{2}\\ pq=-2\end{cases}$ Using $(p+q)^2=p^2+q^2+2pq,$ $x^2=2y+2\times-2$ $\therefore$ locus of M is $x^2=2y-4.$	2 marks : correct solution 1 mark : significant progress towards correct solution

PE5	d) (i) $\frac{d\left(xe^{2x}\right)}{dx} = x.2e^{2x} + e^{2x}.1$ $\therefore \frac{d\left(xe^{2x}\right)}{dx} = 2xe^{2x} + e^{2x}$ (ii) Since $\frac{d\left(xe^{2x}\right)}{dx} = 2xe^{2x} + e^{2x}$ Then $\int\left(2xe^{2x} + e^{2x}\right)dx = xe^{2x} + c$ $\therefore \int 2xe^{2x}dx + \int e^{2x}dx = xe^{2x} + c$ $\int 2xe^{2x}dx + \frac{1}{2}e^{2x} = xe^{2x} + c$ $\therefore \int 2xe^{2x}dx = xe^{2x} - \frac{1}{2}e^{2x} + c$ Hence, $\therefore \int_0^1 2xe^{2x} = \left[xe^{2x} - \frac{1}{2}e^{2x}\right]_0^1$ $\therefore \int_0^1 2xe^{2x} = e^2 - \frac{1}{2}e^2 - \left(0 - \frac{1}{2}\right)$ $\therefore \int_0^1 2xe^{2x} = \frac{1}{2}e^2 + \frac{1}{2}$	1 mark : correct answer 2 marks : correct solution 1 mark : significant progress towards correct solution

Year 12	Mathematics	Half Yearly Examination 2013
Question 12	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

H8 uses techniques of integration to calculate areas and volumes

Outcomes	**Working**	**Marks**
H8	**<u>Question 10:</u>** $(a)\ \int_{-1}^{3} y\,dx = \int_{-1}^{3}(1+x)^{\frac{1}{2}}\,dx$ $= \frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_{-1}^{3}$ $= \frac{2}{3}\{4^{\frac{3}{2}} - 0\}$ $= \frac{16}{3}U^2$ 	2 marks for complete solution. 1 mark for satisfactory working. Diagram not essential but useful.
H8	$(b)\ (i)$ Area $= \frac{3}{6}\{0 + 4\times\sqrt{6\cdot 75} + 2\times 3 + 4\times\sqrt{6\cdot 75} + 0\}$ $= \frac{1}{2}\{8\sqrt{6.75} + 6\}$ $= 13\cdot 4U^2$ (to 3 significant figures)	2 marks for full solution. 1 mark for correct formula
	(ii) Area of semicircle $= \frac{\pi \times 3^2}{2}$ Hence $13\cdot 4 \approx \frac{9\pi}{2}$ and $\pi \approx 2\frac{44}{45} \approx 2\cdot 98$ (2 significant figures)	1 mark
	(iii) Error $\approx 0\cdot 164$ (3 significant figures) Percentage error $\approx \frac{0\cdot 164}{\pi}\times 100$ $\approx 5\cdot 2\%$	1 mark
	Note: Using the full values (without rounding) approx error is $5\cdot 3\%$	
H8	$(c)\ (i)\ 2x^2 = 6 - x^2$ gives us $x = -\sqrt{2}$ and $x = \sqrt{2}$	1 mark

H8	(*ii*) We need to find the volume of revolution between $y = 6 - x^2$ and $y = 2x^2$ from $y = 0$ to $y = 4$ and between $y = 2x^2$ and $y = 6 - x^2$ from $y = 4$ to $y = 6$ i.e. $\left[\int_0^4 \pi(6-y)dy - \int_0^4 \pi\frac{y}{2}dy\right] + \left[\int_4^6 \pi\frac{y}{2}dy - \int_4^6 \pi(6-y)dy\right]$ $= \pi\left(\left[\{6y - \frac{y^2}{2}\}_0^4 - \{\frac{y^2}{4}\}_0^4\right] + \left[\{\frac{y^2}{4}\}_4^6 - \{6y - \frac{y^2}{2}\}_4^6\right]\right)$ $= \pi\left(\left[\{24-8\} - \{4-0\}\right] + \left[\{9-4\} - \{18-16\}\right]\right)$ $= 15\pi U^3$	1 mark for substantial progress 2 marks for full solution.