JAMES RUSE AGRICULTURAL HIGH SCHOOL
TERM 1 ASSESSMENT 1997
YEAR 12 3/4 UNIT

Time: 85 minutes.

Hand in each question separately

QUESTION 1 (10 marks)

(a) After investing \$P for 6 years at a rate of 5% per annum compounded annually, the amount matured to \$268. Find P to the nearest dollar.

(b) Differentiate with respect to x:

(i) $y = \cos(x^2)$

(ii) $y = \log_e(\sin 2x)$

(c) Find the equation of the tangent to the curve $y = \operatorname{cosec} x$ at $x = \frac{\pi}{6}$.

QUESTION 2 (10 marks)

(a) Find the general solution to: $\cos 2x - 1 = 0$.

(b) Use the table of integrals provided to find:

$$\int \left(\frac{\sec 4x \tan 4x}{4}\right) dx$$

(c) Evaluate $\cos^{-1}\left(-\frac{1}{2}\right)$ exactly (answer in radians).

(d) Differentiate with respect to x:

$$y = \cos^{-1}\left(\frac{x}{4}\right)$$

(e) Without the use of calculus, find the minimum value of $(\tan^{-1}x + \sin x)$ for $0 \le x \le 1$. Justify your answer.

QUESTION 3 (10 marks)

(a) Evaluate: (i) $\int_0^2 \frac{dx}{4 + x^2}$

(ii) $\int_0^2 \frac{4x}{4 + x^2}\, dx$

(b) (i) Draw a neat sketch of: $y = \frac{\pi}{2} + 2\sin^{-1}\left(\frac{x}{2}\right)$.

(ii) State its DOMAIN and RANGE.

QUESTION 4 (10 marks)

(a) (i) Express $(\sin x + \sqrt{3}\cos x)$ in the form $A\cos(x - \alpha)$ for $A > 0$ and α acute.

(ii) Find the maximum value of $(\sin x + \sqrt{3}\cos x)$.

(b) Evaluate: $\int_0^{\frac{\pi}{12}} (1 + \sin^2 2x)\, dx$

(c) Given that $f(x) = 2 - (x + 1)^2$ is defined for $x \ge -1$ and has an inverse $f^{-1}(x)$, neatly sketch the graph of $y = f^{-1}(x)$.

QUESTION 5 (10 marks)

(a) Draw a neat sketch of $y = \cos(\sin^{-1}x)$, clearly showing all x,y intercepts.

(b) John takes out a \$25,000 loan at 9% per annum, the interest compounded monthly, and agrees to pay the loan by equal monthly instalments over six years.

(i) Show that each monthly repayment is approximately \$450.

(ii) Assuming he repays \$450 per month, how much is still owed after making the 24th repayment?

(iii) After the 24th repayment he makes a lump sum payment of \$10,000. If he wishes to still make 48 more repayments, what would be the new value of each repayment?

QUESTION 6 (10 marks)

(a) The area bounded between the co-ordinate axes, the curve $y = \frac{1}{\sqrt{1 + 4x^2}}$ and the line $x = \frac{1}{2}$ is rotated about the x-axis.

(i) Find the volume generated by the rotated area.

(ii) Determine the volume as x increases indefinitely.

(b) Given that $y_1 = 2\tan^{-1}\sqrt{x}$ and $y_2 = \sin^{-1}\left(\frac{x - 1}{x + 1}\right)$ for $x > 0$,

(i) evaluate $\frac{d}{dx}(y_1 - y_2)$.

(ii) If $f(x) = y_1 - y_2$, deduce that $f(x)$ is a constant, and find the value of the constant.

(iii) Draw a neat sketch of the function $f(x)$.

END OF PAPER

TERM 1 ASSESSMENT YEAR 12
3/4 UNIT 1997 SOLUTIONS

1 (a) $268 = P(1.05)^6$

$\therefore P = \$200$ 2

(b)

(i) $\frac{dy}{dx} = -2x\sin(x^2)$ 2

(ii) $\frac{dy}{dx} = \frac{1}{\sin 2x}\cdot 2\cos 2x$

$= 2\cot 2x$ 2

(c) $\frac{dy}{dx} = -\operatorname{cosec}x\cot x$

$= -\operatorname{cosec}\frac{\pi}{6}\cot\frac{\pi}{6}$ at $x = \frac{\pi}{6}$

Let $m = -2\sqrt{3}$

Equation of tangent is:

$y - 2 = -2\sqrt{3}(x - \frac{\pi}{6})$

$= -2\sqrt{3}x + \frac{\sqrt{3}\pi}{3}$

$\therefore y = -2\sqrt{3}x + (\frac{\sqrt{3}}{3}\pi + 2)$ 4

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2 (a) $\cos 2x = 1 = \cos 0$

$\therefore 2x = 2n\pi$

$\therefore x = n\pi$ 2

(b) $\frac{1}{16}\sec 4x + C$ 2

(c) $\pi - \cos^{-1}(\frac{1}{2})$

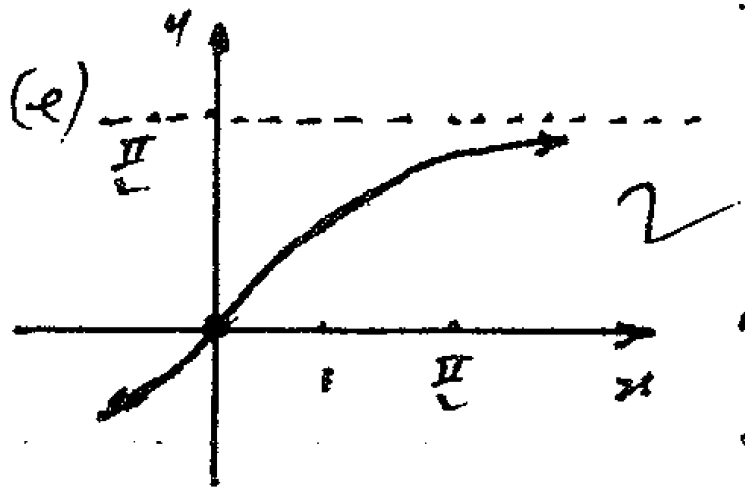
$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$ 2

(d) $\frac{dy}{dx} = -\frac{1}{\sqrt{1-\frac{x^2}{16}}}\times\frac{1}{4}$

$= -\frac{1}{\sqrt{16-x^2}}$ 2

(e)



2   Since both functions are increasing and $f(0)=0$

$\therefore$ MIN at $x=0$

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3 (a)(i) $\int_0^2 \frac{dx}{4+x^2}$

$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$

$= \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0)$

$= \frac{\pi}{8}$ 3

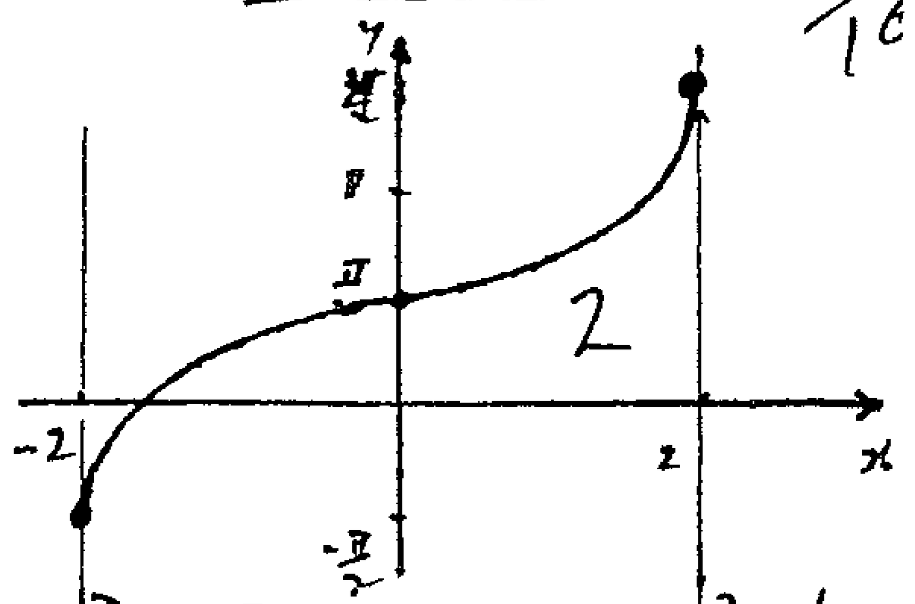
(ii) $\int_0^2 \frac{4x}{4+x^2}\,dx$

$= 2\left[\ln(4+x^2)\right]_0^2$

$= 2(\ln 8 - \ln 4)$

$= 2\ln 2$ 3

(b)(i)



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$D = \{x : -2 \le x \le 2\}$ 1

$R = \{y : -\frac{\pi}{2} \le y \le \frac{3\pi}{2}\}$ 1

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4 (a)(i) $\sin x + \sqrt{3}\cos x$

$= A(\cos x\cos\alpha + \sin x\sin\alpha)$

$\Rightarrow A\cos\alpha = \sqrt{3}$ $A\sin\alpha = 1$

$\therefore \alpha = \frac{\pi}{6}$ $A = 2$

$\therefore \sin x + \sqrt{3}\cos x$

$= 2\cos(x - \frac{\pi}{6})$ 2

(ii) Max. value is 2   1

(b) $\int_0^{\frac{\pi}{12}}(1+\sin^2 2x)\,dx$

$= \int_0^{\frac{\pi}{12}}\left[1 + \frac{(1-\cos 4x)}{2}\right]dx$

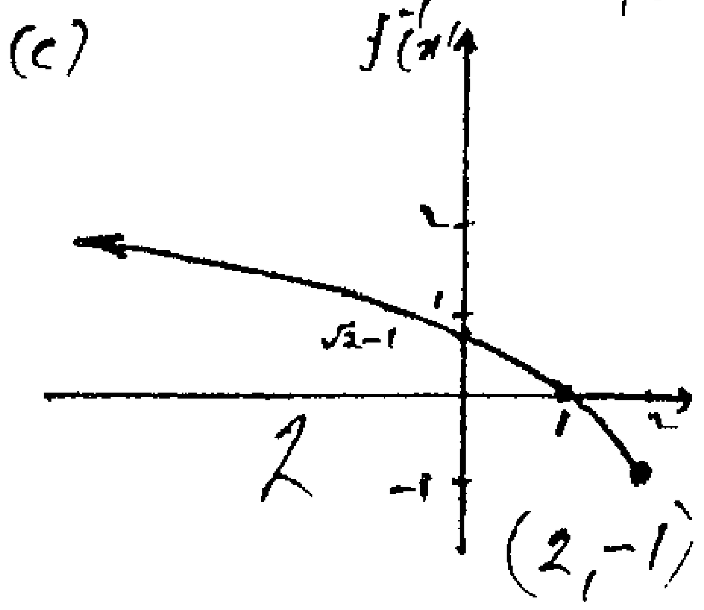
$= \frac{3}{2}\int_0^{\frac{\pi}{12}}dx - \frac{1}{2}\int_0^{\frac{\pi}{12}}\cos 4x\,dx$

$= \frac{3}{2}[x]_0^{\frac{\pi}{12}} - \frac{1}{8}[\sin 4x]_0^{\frac{\pi}{12}}$

$= \frac{3}{2}\cdot\frac{\pi}{12} - \frac{1}{8}(\sin\frac{\pi}{3} - 0)$

$= \frac{2\pi - \sqrt{3}}{16}$ 4

(c)



2

$D = \{x : x \le 2\}$ 1

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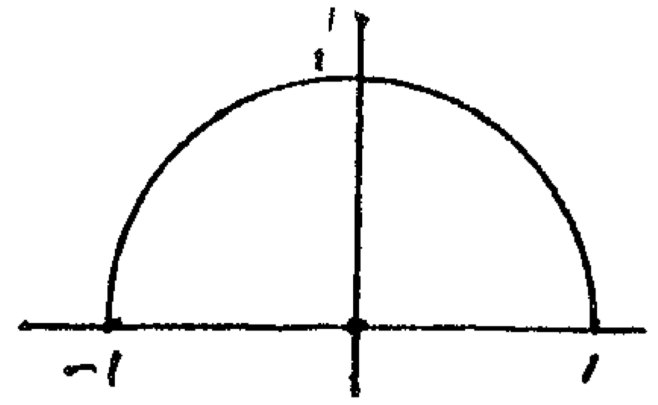
5 (a) $y = \cos(\sin^{-1}x)$ Let $U = \sin^{-1}x$ $x = \sin U$

$y = \cos U$ $-1 \le x \le 1$ $\Rightarrow x^2 + y^2 = 1$

$0 \le y \le 1$ $-\frac{\pi}{2} \le U \le \frac{\pi}{2}$ for $y \ge 0$ 3



(b) (i) Let the amount be \$A.

	B	E
1	$25000\left(\frac{403}{400}\right)$	$25000\left(\frac{403}{400}\right) - A$
2	$25000\left(\frac{403}{400}\right)^2 - A\left(\frac{403}{400}\right)$	$25000\left(\frac{403}{400}\right)^2 - A\left(\frac{403}{400}\right) - A$
3	$25000\left(\frac{403}{400}\right)^3 - A\left(\frac{403}{400}\right)^2 - A\left(\frac{403}{400}\right)$	$25000\left(\frac{403}{400}\right)^3 - A\left(\frac{403}{400}\right)^2 - A\left(\frac{403}{400}\right) - A$

at the end of six years, i.e. 72 months:

$25000\left(\frac{403}{400}\right)^{72} = A\left[\left(\frac{403}{400}\right)^{71} + \left(\frac{403}{400}\right)^{70} + \cdots + 1\right]$

$\therefore A = 25000\left(\frac{403}{400}\right)^{72} \div \frac{\left[\left(\frac{403}{400}\right)^{72} - 1\right]}{\left(\frac{403}{400}\right) - 1}$

$\therefore A = \$450$ 3   (\$450.64)

(ii) after the 24th repayment

Let $D = 25000\left(\frac{403}{400}\right)^{24} - 450\frac{\left[\left(\frac{403}{400}\right)^{24} - 1\right]}{\left(\frac{3}{400}\right)}$

$\therefore D = \$18,125.53$ 2

(iii) Let $P = \$18126 - \$10,000 = \$8126$

$\therefore 8126\left(\frac{403}{400}\right)^{48} = A\left[1 + \left(\frac{403}{400}\right) + \left(\frac{403}{400}\right)^2 + \cdots + \left(\frac{403}{400}\right)^{47}\right]$

$\therefore A = \$202.20$ 2

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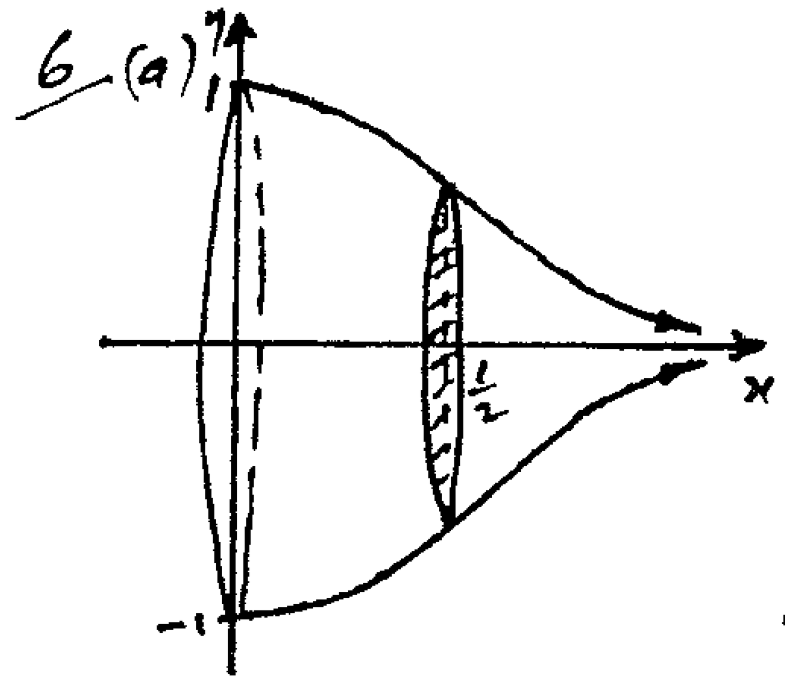
6 (a)



(i) $V = \pi\int_0^{\frac{1}{2}}\left(\frac{1}{\sqrt{1+4x^2}}\right)^2 dx$

$= \frac{\pi}{4}\int_0^{\frac{1}{2}}\frac{1}{(\frac{1}{2})^2 + x^2}\,dx$

$= \frac{\pi}{2}\left[\tan^{-1}2x\right]_0^{\frac{1}{2}}$

$\therefore V = \frac{\pi^2}{8}\,u^3$ 3

(ii) Now, let

$V = \frac{\pi}{2}\left[\tan^{-1}2x\right]_0^x$

As $x \to \infty$, $\tan^{-1}2x \to \frac{\pi}{2}$

$\therefore V = \frac{\pi^2}{4}\,u^3$ 2

(b)(i) $\frac{d}{dx}(y_1 - y_2) = \frac{2}{(1+x)}\cdot\frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\cdot\left[\frac{(x+1)-(x-1)}{(x+1)^2}\right]$

$= \frac{1}{(1+x)\sqrt{x}} - \sqrt{\frac{(x+1)^2}{4x}}\times\frac{2}{(x+1)^2}$

$= \frac{1}{(1+x)\sqrt{x}} - \frac{1}{(1+x)\sqrt{x}}$ 3 $\left(\sqrt{(x+1)^2} = x+1 \text{ since } x > 0\right)$

$\therefore \frac{d}{dx}f(x) = 0$

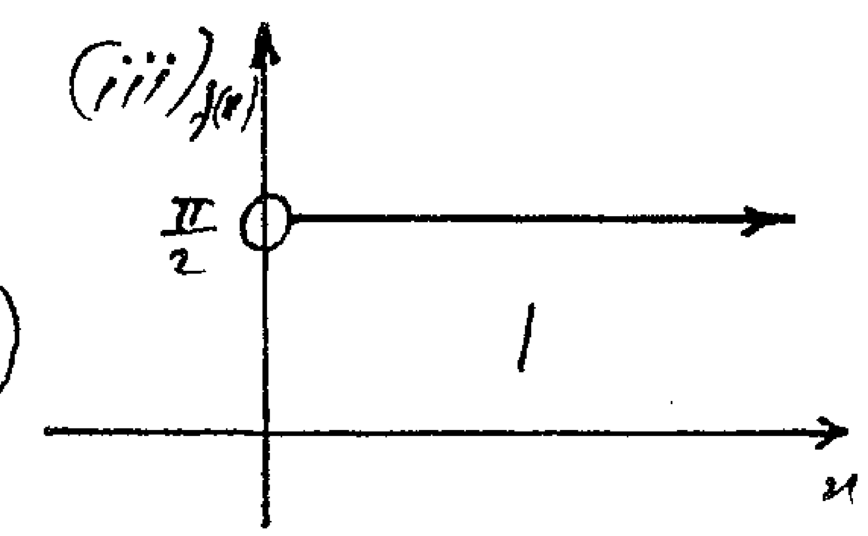
(ii) Since $\frac{d}{dx}f(x) = 0$ $\therefore f(x) = $ Constant

and since $x > 0$, let $x = 1$ 1

$\therefore f(x) = 2\tan^{-1}1 + \sin^{-1}0 = \frac{\pi}{2}$

$\therefore f(x) = \frac{\pi}{2}$ for $x > 0$

(iii)



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