JAMES RUSE AGRICULTURAL HIGH SCHOOL
YEAR 12 ASSESSMENT TERM 1 1999
MATHEMATICS 3/4 UNIT

Time: 85 Minutes
All questions to be attempted.
Approved calculators may be used.
All necessary working must be shown.
All questions are of equal value.
Each section is to be handed in separately.

QUESTION 1 (Start a new page)

(a) Find $\int \sqrt{\sin x} \cos x \, dx$

(b) Differentiate with respect to x:

(i) $y = x \sin 2x$

(ii) $y = \ln \cos x$

(iii) $y = \dfrac{\cos x}{\sin x + 1}$

(c) Find the volume of the solid generated by revolving about the x-axis the region bounded by the co-ordinate axes, the curve $y = \sec x$ and the line $x = \dfrac{\pi}{4}$.

QUESTION 2 (Start a new page)

(a) Find the exact value of the following definite integrals.

(i) $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-4x^2}}$

(ii) $\displaystyle\int_{-\sqrt{3}}^{0} \frac{4\,dx}{1+x^2}$

(b) (i) For what set of values of x will the infinite geometric series $1 - 2x + 4x^2 - 8x^3 + \ldots$ have a limiting sum?

(ii) If this limiting sum is $\dfrac{3}{5}$, find the value of x.

(c) Find the exact value of $\sec\left[\sin^{-1}\left(-\dfrac{3}{4}\right)\right]$

QUESTION 3 (Start a new page)

(a) Two concentric circles are expanding. At a certain instant the outer radius (R) is 4m and the inner radius (r) is 95cm. The outer radius is expanding at the rate of 1m/s and the inner radius at 0.25m/s. Find the rate of change of area (A) between the circles. Give your answer correct to 2 significant figures.

(b) Find the sum of the first 1000 terms of the series

$1 - 2 + 3 - 4 + 5 - 6 + \ldots + (-1)^{n+1} n + \ldots$

(c) (i) Differentiate $\sin^{-1}(\cos 2x)$

(ii) Hence, or otherwise sketch the graph of $y = \sin^{-1}(\cos 2x)$ for $-\pi \le x \le \pi$

QUESTION 4 (Start a new page)

(a) Integrate using the substitution given

$$\int_0^{\frac{\pi}{4}} \frac{dx}{9\cos^2 x + 25\sin^2 x} \qquad \text{where } u = \tan x$$

(b) Express $\sin(2\cos^{-1} x)$ in terms of x only.

(c)

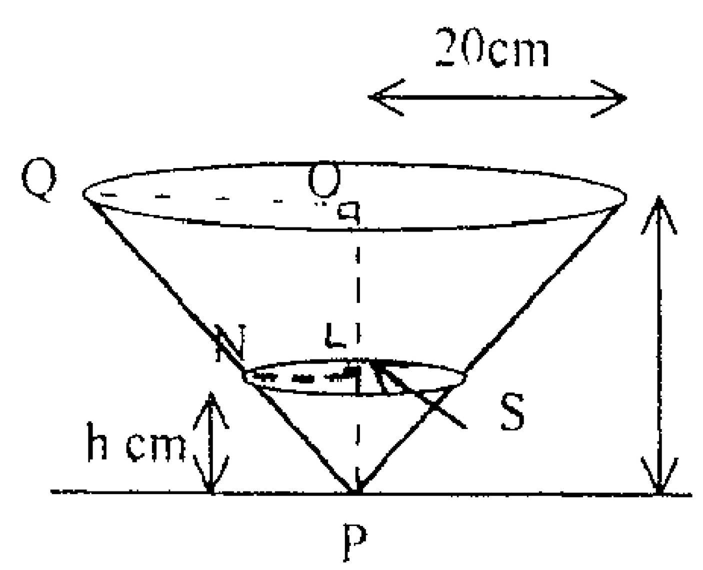

Diagram not to scale

Water is poured into a conical vessel at a constant rate of 10cm^3 per second. The depth of water is h cm at any time t seconds. What is the rate of increase of the area of the surface S of the liquid when the depth is 8cm?

QUESTION 5 (Start a new page)

(a) Katherine borrows $20 000 at 8% per annum reducible interest, calculated monthly. The loan is to be repaid in 60 equal monthly instalments.

(i) Show that the monthly repayments should be $405.53

(ii) With the 8th repayment, Katherine pays an additional $2000, so this payment is $2405.53. After this, repayments continue at $405.53 per month. How many more repayments will be needed?

(b) Calculate the exact area of the region bounded by the graph of $f(x) = 2\cos^{-1}3x$, the x axis, and the ordinates $x = 0$ and $x = \frac{1}{6}$.

QUESTION 6 (Start a new page)

(a) (i) Show that $\frac{d}{dx}\left(\cos^{-1}\left(\frac{1}{x}\right)\right)$ is always positive

(ii) State the domain and range of $y = \cos^{-1}\left(\frac{1}{x}\right)$

(iii) Hence, sketch the graph of $y = \cos^{-1}\left(\frac{1}{x}\right)$ for $-2\pi \le x \le 2\pi$

(b) Two circles of unit radius intersect at A and B respectively. If their centres P and Q are 2x units apart, show that the area (A) common to the two circles is given by:

$$A = 2[\cos^{-1}x - x\sqrt{1-x^2}]$$

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x > 0$

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QUESTION 1:

(a) $\int \sqrt{\sin x} \cdot \cos x \, dx = \frac{2}{3} \sin^{\frac{3}{2}} x + c$ (2)

(b) (i) $y = x \sin 2x$

$\frac{dy}{dx} = x \times 2\cos 2x + \sin 2x \times 1$

$= 2x \cos 2x + \sin 2x$ (2)

(ii) $y = \ln(\cos x)$

$\frac{dy}{dx} = \frac{1}{\cos x} \times -\sin x$

$= -\tan x$ (2)

(iii) $y = \frac{\cos x}{\sin x + 1}$

$\frac{dy}{dx} = \frac{(\sin x + 1) \times (-\sin x) - \cos^2 x}{(\sin x + 1)^2}$

$= \frac{-1 - \sin x}{(\sin x + 1)^2}$

$= \frac{-1}{(\sin x + 1)}$ (2)

(c) $V = \pi \int_0^{\frac{\pi}{4}} \sec^2 x \, dx$

$= \pi \left[\tan x\right]_0^{\frac{\pi}{4}}$

$V = \pi [1 - 0]$

$V = \pi \, u^3$ (2)

10

QUESTION 2:

(a) (i) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\left[\sin^{-1} 2x\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\left[\sin^{-1} 1 - \sin^{-1} 0\right]$

$= \frac{\pi}{4}$ (2)

(ii) $\int_{-\sqrt{3}}^{0} \frac{4\,dx}{1+x^2} = 4\left[\tan^{-1} x\right]_{-\sqrt{3}}^{0}$

$= 4\left[\tan^{-1} 0 - \tan^{-1}(-\sqrt{3})\right]$

$= \frac{4\pi}{3}$ (2)

(b) (i) $-1 < r < 1$

$-1 < -2x < 1$

$\therefore -\frac{1}{2} < x < \frac{1}{2}, \; x \neq 0$ (2)

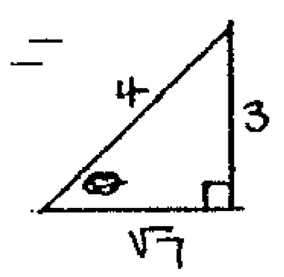
(ii) $\frac{3}{5} = \frac{1}{1+2x}$

$5 = 3 + 6x$

$x = \frac{1}{3}$ (2)

(c) $\sec\left[\sin^{-1}\left(-\frac{3}{4}\right)\right]$ (2)

$\therefore \sec\theta = \frac{4}{\sqrt{7}}$



Let $\theta = \sin^{-1}\left(-\frac{3}{4}\right)$

$\sin\theta = -\frac{3}{4}$

$\therefore \sec\left[\sin^{-1}\left(-\frac{3}{4}\right)\right] = \frac{4}{\sqrt{7}}$

$= \frac{4\sqrt{7}}{7}$

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QUESTION 3:

(a) $A = \pi(R^2 - r^2)$

$\frac{dA}{dt} = 2\pi\left(R \cdot \frac{dR}{dt} - r \cdot \frac{dr}{dt}\right)$

$= 2\pi(4 \times 1 - 0.95 \times 0.25)$

$= 2\pi(3.7625)$

$= 23.64048 \ldots$

$= 24 \text{ m}^2/\text{s}$ (2 s.f.) (3)

(b) $1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n-1} n + \cdots$

$S_{1000} = 500(-1)$

$= -500$ (1)

(c) $\frac{d}{dx}\left[\sin^{-1}(\cos 2x)\right]$

Let $y = \sin^{-1} u$ where $u = \cos 2x$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot -2\sin 2x$

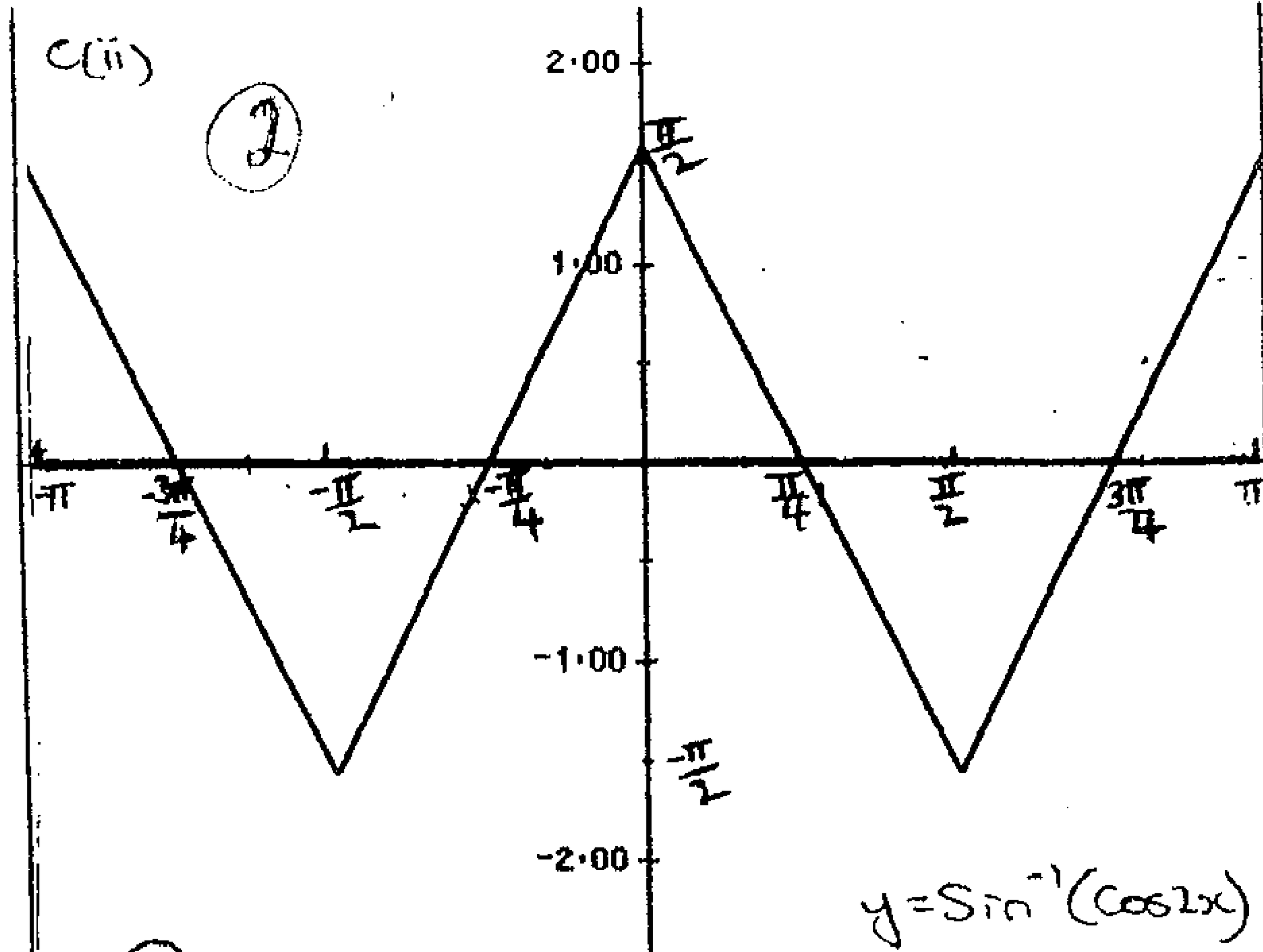
$= \frac{-2\sin 2x}{\sqrt{1-\cos^2 2x}}$

$= \frac{-2\sin 2x}{|\sin 2x|}$ (3)

$= -2$ if $\sin 2x > 0$

2 if $\sin 2x < 0$

c(ii) (2)



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QUESTION 4:

(a) $\int_0^{\frac{\pi}{4}} \frac{dx}{9\cos^2 x + 25\sin^2 x}$

$u = \tan x$

$du = \sec^2 x \, dx$

when $x = 0$ $u = 0$

$x = \frac{\pi}{4}$ $u = 1$

[1] $= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x \, dx}{9 + 25\tan^2 x}$ ($\div$ by $\cos^2 x$)

$= \int_0^1 \frac{du}{9 + 25u^2}$

$= \frac{1}{25}\int_0^1 \frac{du}{u^2 + \frac{9}{25}}$ (3)

[1] $= \frac{1}{25} \times \frac{5}{3}\left[\tan^{-1}\frac{5u}{3}\right]_0^1$

[1] $= \frac{1}{15}\tan^{-1}\frac{5}{3}$ OR $\doteqdot 3.94$

(b) $\sin(2\cos^{-1} x)$

Let $\cos^{-1} x = \alpha$

$\therefore \cos\alpha = x$

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$

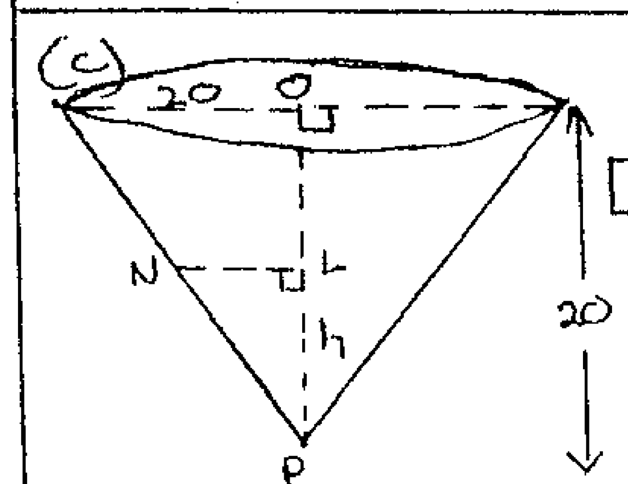
$\sin\alpha = \sqrt{(1 - \cos^2\alpha)}$

$= \sqrt{1 - x^2}$ (3)

$\therefore 2\sin\alpha\cos\alpha = 2\left[\sqrt{1-x^2}\right] \cdot x$

$\therefore \sin(2\cos^{-1} x) = 2x\sqrt{1-x^2}$

(c)



In $\triangle OQP$ and $\triangle LNP$

$\angle OPQ$ is common

[1] $\angle QOP = \angle LNP$ (corresp. angles $OQ \parallel LN$)

$\therefore \triangle PLN \parallel\parallel \triangle POQ$ (equiangular)

$\therefore \triangle PLN$ is isosceles

[1] $\therefore LN = LP = h$

Now $V = \frac{1}{3}\pi h^3$

$S = \pi h^2$

$\frac{dV}{dt} = \pi h^2 \cdot \frac{dh}{dt}$ $\quad \frac{dV}{dt} = 10$

$\therefore \frac{dh}{dt} = \frac{10}{\pi h^2}$

at $h = 8$

$\frac{dh}{dt} = \frac{5}{32\pi}$ cm/s [1]

$\frac{dS}{dt} = 2\pi h \cdot \frac{dh}{dt}$

$\left[\frac{dS}{dt}\right]_{h=8} = 2\pi \times 8 \times \frac{5}{32\pi}$

$\therefore \frac{dS}{dt} = \frac{5}{2}$ cm²/s [1]

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QUESTION 5:

(a)(i) $M = \frac{20000 \times 1.006^{60} \times 0.006}{1.006^{60} - 1}$

[2] $= \$405.53$ (nearest cent)

(ii) after 8th normal payment, amt. owing is:

$A = 20000 \times (1\frac{1}{150})^8 - \$405.53 \frac{((1\frac{1}{150})^8 - 1)}{(1\frac{1}{150} - 1)}$

$\therefore A = \$17770.93$ [1]

Repays \$2000

$\therefore$ Amt. owing $= \$15770.93$

$0 = 15770.93 \times (1\frac{1}{150})^n - 405.53 \frac{((1\frac{1}{150})^n - 1)}{\frac{1}{150}}$

[1] $= 15770.93 \times (1\frac{1}{150})^n - 60829.50((1\frac{1}{150})^n - 1)$

$= (1\frac{1}{150})^n \left[15770.93 - 60829.5\right] + 60829.5$

$= (1\frac{1}{150})^n \left[-45058.57\right] + 60829.5$

$\therefore \frac{60829.5}{45058.57} = (1\frac{1}{150})^n$ [1]

$1.3500096 = (1\frac{1}{150})^n$

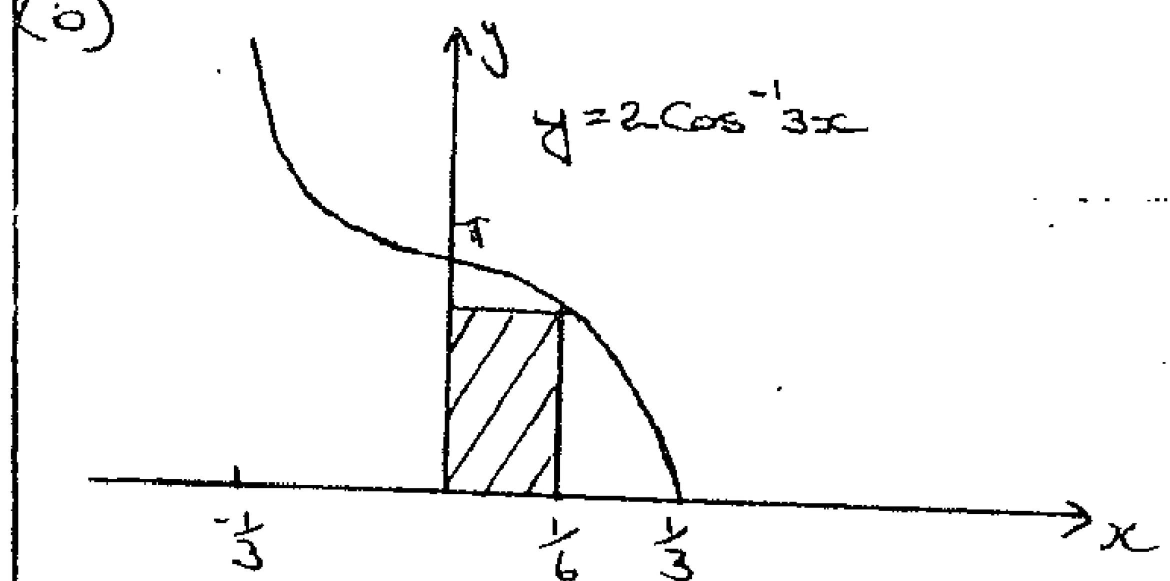
$\therefore n = 45.17$

[1] $n = 46$ repayments

(4)

QUESTION 5 (cont.)

(a)



$A = \int_0^{\frac{1}{6}} (2\cos^{-1}3x)\,dx$

Let $y = 2\cos^{-1}3x$

$\frac{y}{2} = \cos^{-1}3x$

$3x = \cos\frac{y}{2}$

$x = \frac{1}{3}\cos\frac{y}{2}$ [1]

$A = \text{Area of Rectangle} + \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{3}\cos\frac{y}{2}\,dy$ [1]

$= \frac{\pi}{9} + \left[\frac{2}{3}\sin\frac{y}{2}\right]_{\frac{2\pi}{3}}^{\pi}$ [1]

$= \frac{\pi}{9} + \frac{2}{3}\left(1 - \frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{9} + \frac{2-\sqrt{3}}{3}$ [1]

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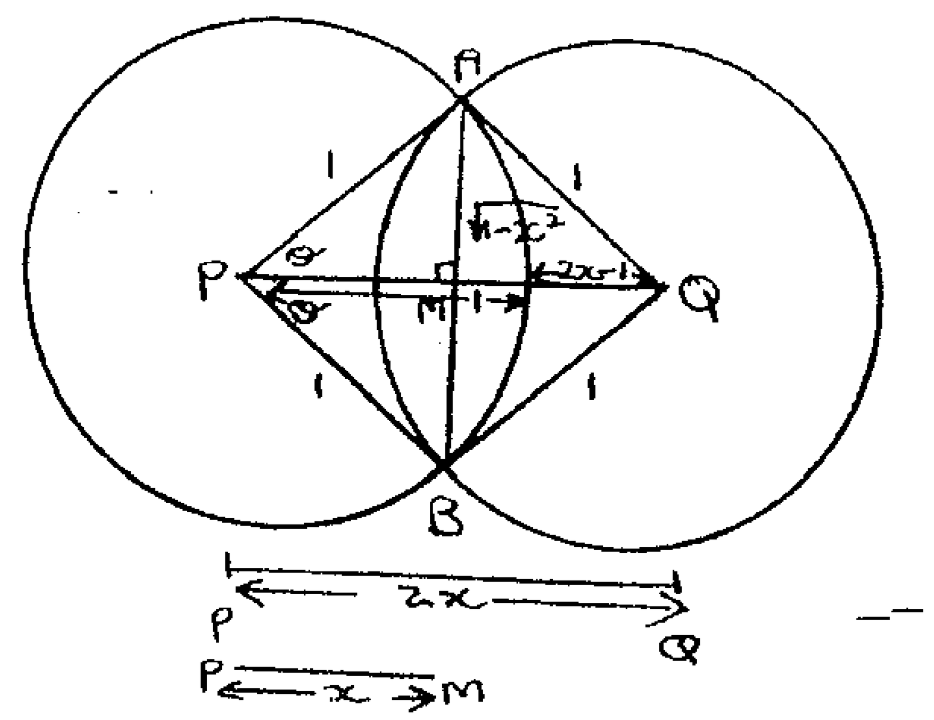
(b)



$A = 2 \times \frac{1}{2} r^2 (2\theta - \sin 2\theta)$

$A = r^2(2\theta - \sin 2\theta)$ $\quad \cos\theta = r$

$A = r^2(2\theta - (2\sin\theta\cos\theta))$

$A = r^2(2\theta - 2\sqrt{1-x^2} \times x)$ $\quad r = 1$

$A = 2(\cos^{-1}x - x\sqrt{1-x^2})$

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QUESTION 6:

(a)(i) $\frac{d}{dx}\left(\cos^{-1}\left(\frac{1}{x}\right)\right) = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{x^2}$

$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$

$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$

$= \frac{1}{\sqrt{x^4-x^2}} > 0$

2

(ii) Domain: $-1 \leq \frac{1}{x} \leq 1$

$\therefore x \leq -1$ and $x \geq 1$ 1

Range: $0 \leq y \leq \pi$ but $y \neq \frac{\pi}{2}$ 1

(iii)

3