James Ruse AHS Year 12 Mathematics Extension 1 Term 1 2001

- **Time allowed 85 minutes.**
- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for carelessly or badly arranged work.
- Standard integrals are printed on page 4.
- Answer each question on a new page.

Question 1 — Marks

(a) Differentiate: 5

(i) $\ln(1+e^x)$

(ii) $\ln(\frac{2x+1}{3x+2})$

(iii) $\frac{e^{3x}}{x^2}$.

(b) Find the indefinite integrals of: 5

(i) $e^{\frac{-x}{a}}$ where a is constant.

(ii) $\frac{x^3}{2-x^2}$

Question 2 Start a new page.

(a) (i) If $y=\tan 3x$ find the value of $\frac{dy}{dx}$ when $x=\frac{\pi}{3}$. 3

(ii) Hence, find the equation of the tangent to the curve $y=\tan 3x$ at the point $(\frac{\pi}{3},0)$

(b) If $f(x)=(ax+b)\sin x+(cx+d)\cos x$, determine the values of the constants $a,b,c\,\&\,d$ such that $f'(x)=x\cos x$. 4

(c) (i) Differentiate $x\tan x$ with respect to x 3

(ii) Hence find $\int x\sec^2 x dx$

Question 3 Start a new page. — Marks

(a) A filter is in the shape of an inverted right circular cone of base radius 2cm and altitude 3cm. If water is flowing out of the bottom at a rate of 5cm^3/min, find the exact rate at which level of the water is falling when the depth is 2cm. 3

(b) Prove by mathematical induction for $n\geq 1$ that: 4

$$1.2^2+2.3^2+3.4^2+\ldots+n(n+1)^2=\frac{1}{12}n(n+1)(n+2)(3n+5)$$

(c) If $f(x)=g(x)-\ln[g(x)+1]$ 3

(i) Prove that $f'(x)=\frac{g(x).g'(x)}{g(x)+1}$.

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}}\frac{\sin x\cos x}{\sin x+1}dx$

Question 4 Start a new page.

(a) Using the fact that $2\cos^2 x=1+\cos 2x$, prove that $8\cos^4 x=3+4\cos 2x+\cos 4x$. 2

(b) (i) Sketch on the same axes, the curves $y=\cos x$ and $y=\cos^2 x$, for $0\leq x\leq\frac{\pi}{2}$. 8

(ii) Find the area enclosed between these curves.

(iii) Find the volume generated when the area from (ii) is rotated about the x axis.

Question 5 Start a new page.

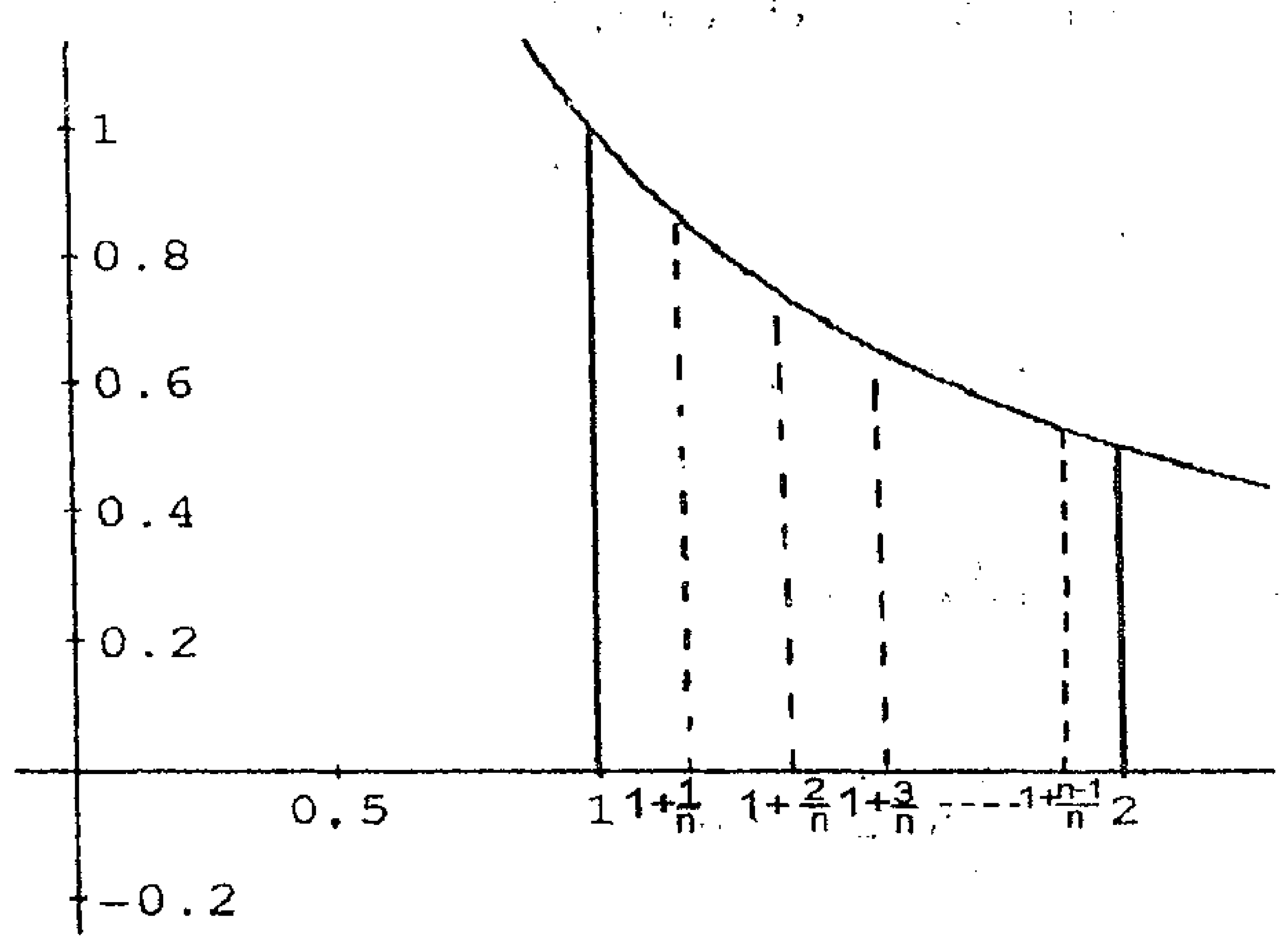
(a) (i) Prove that $\cot x+\tan x=2\operatorname{cosec}2x$. 4

(ii) Hence evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}2\operatorname{cosec}2x dx$.

(b) Given that $a^x=b^y=(ab)^z$, prove that $\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$. 3

Question 5 (cont.) **Marks**

(c)



Consider the curve $y = \frac{1}{x}$ for $x > 0$. Divide the interval from $x = 1$ to $x = 2$ into n equal parts, each of width $\frac{1}{n}$. From the definition of the definite integral show that: **3**

$$\lim_{n \to \infty}\left\{\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right\} = \ln 2$$

Question 6 Start a new page.

(a) Determine the values of k for which $y = e^{kx}$ satisfies the equation **2**

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0.$$

(b) A prize fund is established with a single investment of \$2000 to provide an **annual** prize of \$150. The fund accrues interest at 5% p.a. **paid half yearly**. If the first prize is awarded one year after the fund is established: **8**

(i) Find the amount in the fund account after the first prize is awarded.

(ii) Show that the amount in the fund account after the 6th prize is awarded is approximately \$1660.

(iii) How many prizes can be awarded before the fund is exhausted?

END OF PAPER

STANDARD INTEGRALS

$\int x^n\,dx$	$= \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x}\,dx$	$= \ln x,\ x > 0$
$\int e^{ax}\,dx$	$= \frac{1}{a}e^{ax},\ a \neq 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax,\ a \neq 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax,\ a \neq 0$
$\int \sec^2 ax\,dx$	$= \frac{1}{a}\tan ax,\ a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax,\ a \neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$= \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$= \ln\left(x + \sqrt{x^2-a^2}\right),\ x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$= \ln\left(x + \sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x,\ x > 0$

1 a (i) $\frac{d}{dx}\ln(1+e^x)$

$= \frac{e^x}{1+e^x}$

(ii) $\frac{d}{dx}\ln\left(\frac{2x+1}{3x+2}\right)$

$= \frac{d}{dx}\left(\ln(2x+1) - \ln(3x+2)\right)$

$= \frac{2}{2x+1} - \frac{3}{3x+2}$

$= \frac{1}{(2x+1)(3x+2)}$ 2

(iii) $\frac{d}{dx}\frac{e^{3x}}{x^2} = \frac{x^2.3e^{3x} - e^{3x}.2x}{x^4}$

$= \frac{e^{3x}(3x-2)}{x^3}$ 2

b (i) $\int e^{-\frac{x}{a}}\,dx$

$= -ae^{-x/a} + c$ 2

(ii) $\int \frac{x^3}{2-x^2}\,dx$

$= \int\left(-x + \frac{2x}{2-x^2}\right)dx$ 3

$= \frac{-x^2}{2} - \ln(2-x^2) + c$

2a $y = \tan 3x$

$y' = 3\sec^2 3x$

$= 3\sec^2\pi$ 2

$= 3$ at $x = \frac{\pi}{3}$

∴ tangent $y = 3\left(x - \frac{\pi}{3}\right)$ 1

$y = 3x - \pi$

2b $f(x) = (ax+b)\sin x + (cx+d)\cos x$

$f'(x) = (ax+b)\cos x + a\sin x$ 1

$+ (cx+d).-\sin x + c\cos x$

$= \sin x(a - cx - d) + \cos x(ax + b + c)$

Now $f'(x) = x\cos x$

∴ $a = 1$, $b + c = 0$

$c = 0$ ∴ $b = 0$ 2

$a - d = 0$

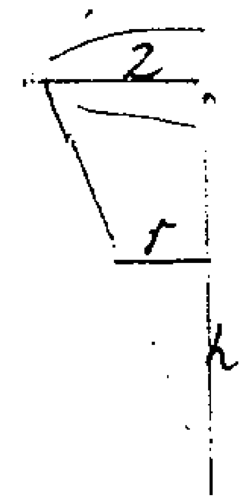
∴ $d = 1$

2c (i) $\frac{d}{dx}\tan x = x\sec^2 x + \tan x$ 1

(ii) $\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$

$= x\tan x - \ln\cos x + c$ 2

Q3 (a)



$\frac{r}{2} = \frac{h}{3}$

$r = \frac{2h}{3}$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi.\frac{4h^2}{9}.h$ 3

$= \frac{4\pi}{27}h^3$

$\frac{dV}{dh} = \frac{4\pi}{9}h^2$

$\frac{dh}{dt} = \frac{dh}{dV}.\frac{dV}{dt}$ 1

$= \frac{9}{4\pi h^2}.-5$

$= \frac{-45}{16\pi}$ cm/min

∴ ... at $\frac{45}{...}$ cm/min

Prove true for $n=1$

LHS $= 1.2^2$

RHS $= \frac{1}{12}(1)(2)(3)(8)$

$= 4$

LHS = RHS

Assume true for $n=k$ i.e. assume

$1.2^2 + 2.3^2 + \ldots k(k+1)^2 = \frac{1}{12}k(k+1)(k+2)(3k+5)$

4 Prove true for $n=k+1$ i.e. Prove

$\frac{1}{12}k(k+1)(k+2)(3k+5) + (k+1)(k+2)^2$

$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+8)$

LHS $= \frac{1}{12}k(k+1)(k+2)(3k+5) + \frac{12(k+1)(k+2)^2}{12}$

$= \frac{1}{12}(k+1)(k+2)\left[k(3k+5) + 12(k+2)\right]$

$= \frac{1}{12}(k+1)(k+2)\left[3k^2 + 17k + 24\right]$

$= \frac{1}{12}(k+1)(k+2)(3k+8)(k+3)$

$=$ RHS

Thus if it is true for $n=1$ it is true for $n=2$ & hence $n=3$ etc.

∴ it is true for all n $(n \geq 1)$

3c (i) $f(x) = g(x) - \ln\left[g(x)+1\right]$

$f'(x) = g'(x) - \frac{1}{g(x)+1}.g'(x)$

$= \frac{g'(x)\left[g(x)+1\right] - g'(x)}{g(x)+1}$ 3

$= \frac{g(x)\,g'(x)}{g(x)+1}$

(ii) $\int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{\sin x+1}\,dx$

$= \left[\sin x - \ln(\sin x+1)\right]_0^{\frac{\pi}{2}}$ from (i)

$= 1 - \ln 2$

4 $2\cos x - 1 ...$

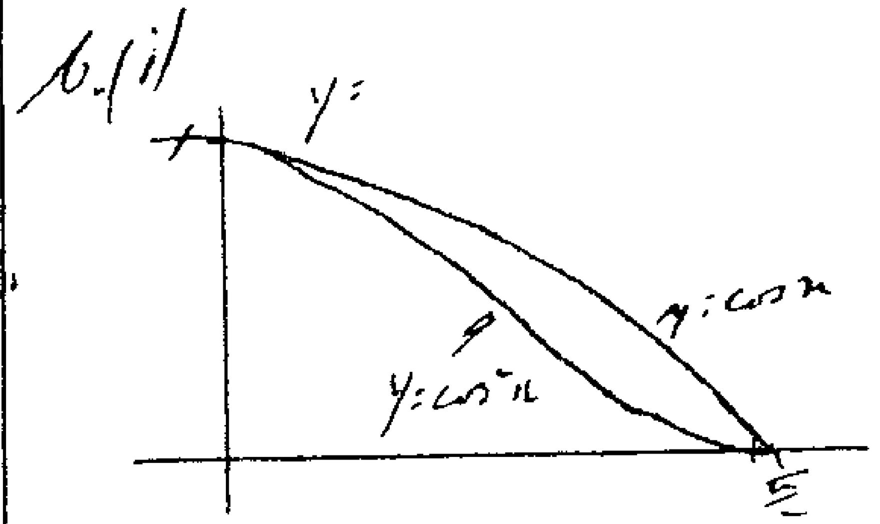
$4\cos^4 x = 1 + 2\cos 2x + \cos^2 2x$

$= 1 + 2\cos 2x + \frac{\cos 4x + 1}{2}$ 2

$8\cos^4 x = 2 + 4\cos 2x + \cos 4x + 1$

$= 3 + 4\cos 2x + \cos 4x$

b (i)



ii $A = \int_0^{\frac{\pi}{2}}(\cos x - \cos^2 x)\,dx$

$= \int_0^{\frac{\pi}{2}} \cos x - \frac{1}{2}(1 + \cos 2x)\,dx$

$= \left[\sin x - \frac{\sin 2x}{2} - \frac{x}{2}\right]_0^{\frac{\pi}{2}}$

$= \left(1 - \frac{\pi}{4}\right)\ u^2$

(iii) $V = \pi\int_0^{\frac{\pi}{2}}(\cos^2 x - \cos^4 x)\,dx$ 1

$= \pi\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\,dx - \left(\frac{3 + 4\cos 2x + \cos 4x}{8}\right)$

$= \pi\left[\left(\frac{x + \frac{\sin 2x}{2}}{2}\right) - \left(\frac{3x + 2\sin 2x + \frac{\sin 4x}{4}}{8}\right)\right]_0$

$= \pi\left[\frac{\pi}{4} - \frac{3\pi}{16}\right]$

$= \frac{\pi^2}{16}\ u^3$

.a (i)

$\cot x + \tan x = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$

$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$

$= \frac{2}{\sin 2x}$

$= 2\,\text{cosec}\, 2x.$

(ii) $\int_{\pi/6}^{\pi/3} 2\,\text{cosec}\, 2x \, dx$

$= \int_{\pi/6}^{\pi/3} [\cot x + \tan x]$

$= [\ln \sin x - \ln \cos x]_{\pi/6}^{\pi/3}$

$= \ln \tan x$

$= \ln\sqrt{3} - \ln\frac{1}{\sqrt{3}} = \ln 3$

b. $a^x = b^y = (ab)^3$

$x \ln a = y \ln b = 3(\ln a + \ln b)$

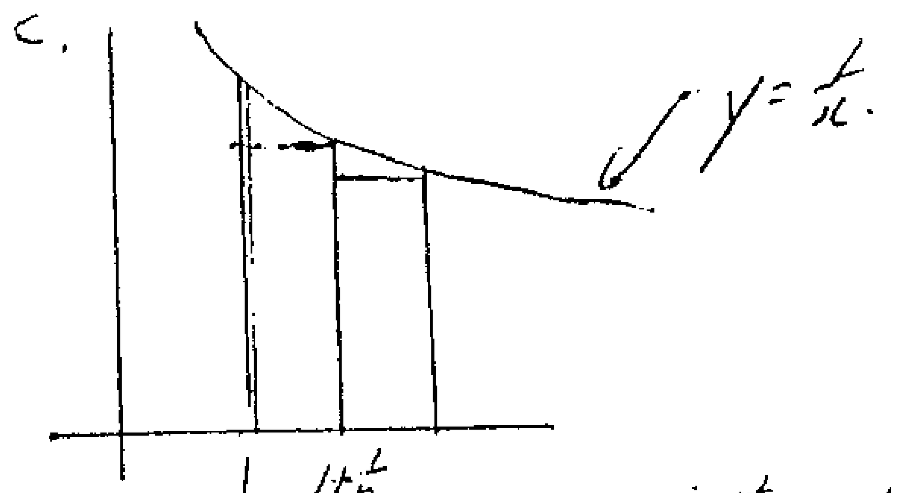
$\frac{1}{x} + \frac{1}{y} = \frac{\ln a}{y \ln b} + \frac{1}{y}$

$= \frac{\ln a + \ln b}{y \ln b}$

$= \frac{\ln a + \ln b}{3(\ln a + \ln b)}$

$= \frac{1}{3}.$ 3

c.



ht of 1st rectangle $\frac{n}{n+1}$ width $= \frac{1}{n}$

ht of 2nd rectangle $= \frac{n}{n+2}$ width $= \frac{1}{n}$

etc.

Summing lower rectangles

$\frac{1}{n+1} + \frac{1}{n+2} \cdots \frac{1}{2n}$

Taking the limit

$A = \lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n}\right)$ 2

By integration $A = \int_1^2 \frac{1}{x}\,dx$

$= [\ln x]_1^2$

$= \ln 2$

2 $\therefore \ln 2 = \lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} \cdots \frac{1}{2n}\right)$ 3

6. $y = e^{kx}$

$\frac{dy}{dx} = ke^{kx}$

$\frac{d^2y}{dx^2} = k^2 e^{kx}$

Now $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$

$k^2 e^{kx} + 7ke^{kx} + 12e^{kx} = 0$

$e^{kx}(k^2 + 7k + 12) = 0$ 3

$\therefore k = -4, -3$

6b

Let F_n be amount in fund after n prizes are awarded.

(i) $\therefore F_1 = 2000 \times 1.025^2 - 150.$ 2

$= \$1951.25$

(ii) ← 6 times →

$F_6 = 2000(1.025)^{12} - 150(1 + 1.025^2 + 1.025^4 \cdots 1.025^{10})$

3 $= 2000\ 1.025^{12} - 150\left(\frac{1.025^{12} - 1}{1.025^2 - 1}\right)$

$= 2689.78 - 1021.89$

$= \$1667.89$

Q6 c.

$F_n = 0 = 2000\ 1.025^{2n} - 150\left(\frac{1.025^{2n} - 1}{0.050625}\right)$

$300\ 1.025^{2n} - 2962.9629(1.025)^{2n} + 2962.96 = 0$

$1.025^{2n} = 3.0769\ldots$

$2n \ln 1.025 = \ln 3.0769\ldots$

$2n = 45.5$

$n = 22$ prizes.