Marks

QUESTION 1 **(10 Marks)**

(a) Find $\int \frac{e^x dx}{4+e^x}$. 1

(b) Use the substitution $x = \frac{3}{2}\sin\theta$ to find the exact value of $\int_0^{3/2} \frac{dx}{\sqrt{9-4x^2}}$. 4

(c) (i) Rewrite the expression $\sqrt{3}\cos x - 3\sin x$ in the form $A\cos(x+\alpha)$ where $A > 0$ and $0 < \alpha < \pi/2$. 2

(ii) Hence sketch the graph of the function $y = \sqrt{3}\cos x - 3\sin x$ for $0 \le x \le 2\pi$. 3

QUESTION 2 **(10 Marks)**

(a) The sum of the first ten terms of the following arithmetic series is -440:

$$\log_2 \frac{1}{x} + \log_2 \frac{1}{x^2} + \log_2 \frac{1}{x^3} + \ldots\ldots + \log_2 \frac{1}{x^{10}} \qquad (x > 0)$$

Find the value of x. 3

(b) (i) If $f(x) = \ln(x + \sqrt{x^2+1})$, find $f'(x)$. 1

(ii) Hence evaluate $\int_0^1 \frac{dx}{\sqrt{x^2+1}}$, leaving your answer in exact form. 2

(c) A water tank is generated by rotating the curve $y = \frac{x^4}{16}$ around the y-axis.

(i) Show that, if the tank is filled to a depth of h cm., the volume of water in the tank is given by

$$V = \frac{8\pi h^{\frac{3}{2}}}{3} \quad \text{cm}^3.$$ 2

(ii) Water drains from the tank through a small hole at its base. The rate of change of the volume of water in the tank is proportional to the square root of the water's depth. Show that the water level in the tank falls at a constant rate. 2

Marks

QUESTION 3 **(10 Marks)**

(a) Find the coordinates of the first stationary point of the function $y = e^{-x} \sin x$ for x > 0 and determine its nature. 4

(b) (i) Use the method of mathematical induction to prove that, if p is a positive integer, then $(1+p)^n - 1$ is divisible by p for all positive integers $n \geq 1$. 4

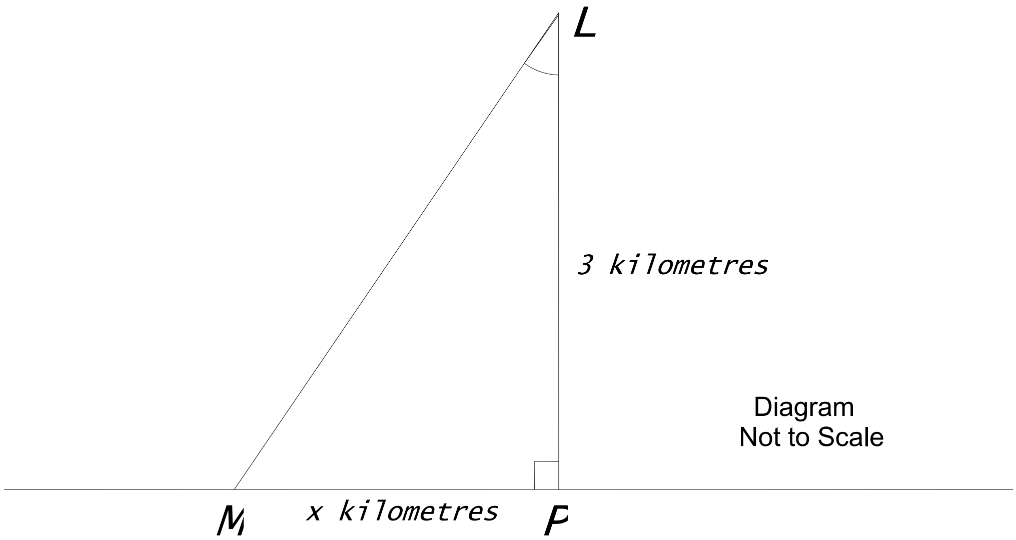
(ii) Hence deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all positive integers $n \geq 1$. 2

QUESTION 4 **(10 Marks)**

(a) (i) Find the exact area enclosed between the curves $y = \cos x$ and $y = \cos^2 x$ for $0 \leq x \leq \frac{\pi}{2}$. 2

(ii) Hence, or otherwise, find the volume of revolution generated when this area is rotated about the *x*-axis. 3

(b)



In the diagram above, a lighthouse *L* containing a revolving beacon is located out at sea, 3 kilometres from *P*, the nearest point on a straight shoreline. The beacon rotates clockwise with a constant rotation rate of 4 revolutions per minute and throws a spot of light onto the shoreline. When the spot of light is at *M*, x km from *P*, the angle at *L* is θ.

(i) Explain why $\frac{d\theta}{dt} = 8\pi$ where t is the time measured in minutes. 1

(ii) How fast is the spot moving when it is at *P*? 2

(iii) How fast is the spot moving when it is at a point on the shoreline 2 km from *P* ? 2

Marks

QUESTION 5 **(10 Marks)**

(a) Find $\dfrac{d}{dx}\ln\left(\dfrac{x^2}{\sqrt{1-x}}\right)$. 2

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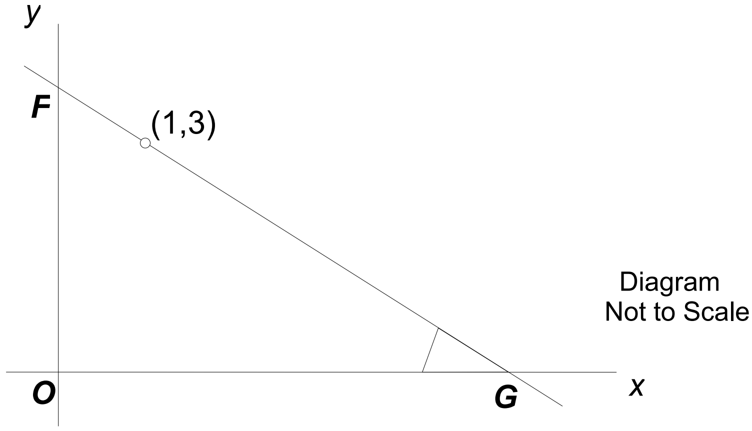


On the set of axes pictured, the point (1,3) lies on the line FG and angle FGO is θ where $0 < \theta < \pi/2$.

(i) Show that the equation of the line FG may be written as
$$y = -(\tan\theta)x + \tan\theta + 3 \ .$$ 2

(ii) Show that the area of ΔOFG can be written as
$$A = \frac{(\tan\theta + 3)^2}{2\tan\theta} .$$ 2

(iii) Find the value of θ for which the area A is a minimum. 4

QUESTION 6 **(10 Marks)**

(a) Find $\displaystyle\int \frac{x-1}{1+x^2}\,\text{dx}$. 2

(b) Upon retirement, a woman invests \$300,000 and receives 6% per annum interest, compounded monthly, on her investment. She withdraws \$2000 as living allowance at the end of each month, immediately after the interest is paid.

(i) Show that the balance in her account after 1 month will be \$299,500. 1

(ii) Show that, after n months, the balance in her account will be given by the expression $\$400{,}000 - (1.005)^n\,\$100{,}000$. 3

(iii) After how long will she run out of money (to the nearest month)? 2

(iv) How much monthly living allowance, to the nearest dollar, should she allow herself if she wishes to plan for a retirement of 30 years ? 2

END OF PAPER

Q1 a) $\ln(4+e^x)+c$ ①

b) $x=\frac{3}{2}\sin\theta \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$dx=\frac{3}{2}\cos\theta\, d\theta$

when $x=0 \quad \theta=0$ ①

$x=3/2 \quad \theta=\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}}\frac{\frac{3}{2}\cos\theta\, d\theta}{\sqrt{9-\frac{9}{4}\sin^2\theta}} = \int_0^{\frac{\pi}{2}}\frac{\frac{\cancel{3}}{2}\cos\theta\, d\theta}{\cancel{3}\sqrt{1-\sin^2\theta}}$$ ①

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}}$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos\theta\, d\theta}{|\cos\theta|}$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{\cos\theta}\, d\theta \quad \left(\text{Since } \cos\theta\ge 0 \text{ for } -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\right)$$ ①

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta$$

$$=\left[\frac{\theta}{2}\right]_0^{\frac{\pi}{2}}$$ ①

$$=\frac{\pi}{4}$$

c) $A\cos(x+\alpha) = A\cos x\cos\alpha - A\sin x\sin\alpha$

$= \sqrt{3}\cos x - 3\sin x$

$\therefore A\cos\alpha=\sqrt{3} \qquad A\sin\alpha = 3$

$\therefore \tan\alpha = \frac{3}{\sqrt{3}}=\sqrt{3} \quad \alpha=\frac{\pi}{3} \quad (0<\alpha<\frac{\pi}{2})$ ①

$(A\cos\alpha)^2 + A(\sin\alpha)^2 = 3+9$

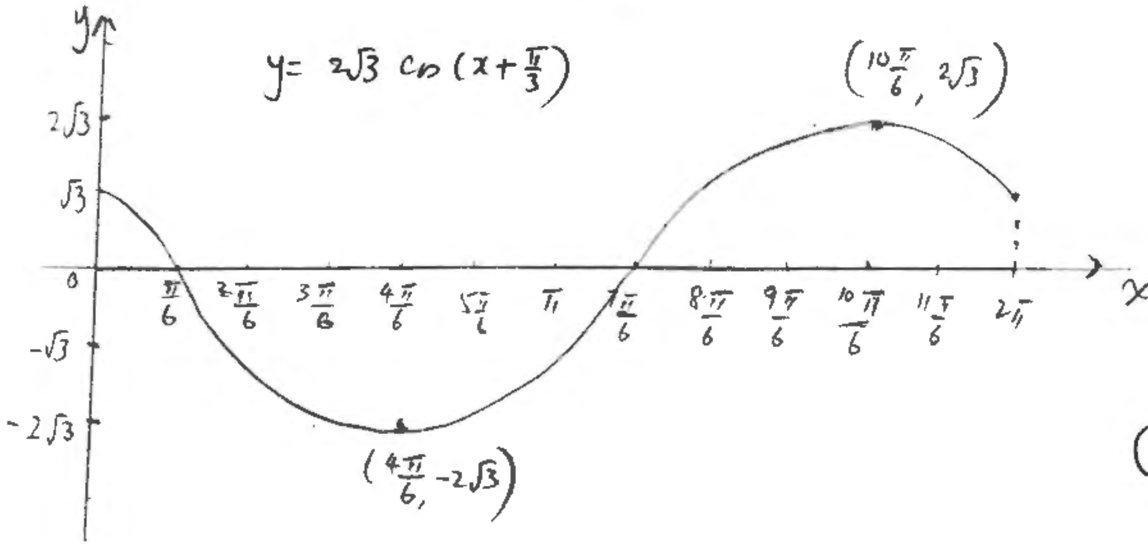
$A^2(\cos^2\alpha + \sin^2\alpha) = 12$

$A^2 = 12$

$A = \sqrt{12} \quad (A > 0)$ ①

$A = 2\sqrt{3}$

$\sqrt{3}\cos x - 3\sin x = 2\sqrt{3}\cos\left(x + \frac{\pi}{3}\right)$



③

Q 2

a) $-[\log_2 x + 2\log_2 x + 3\log_2 x + \cdots + 10\log_2 x] = -440$ ①

$\left(\frac{1+10}{2}\right) 10 \cdot \log_2 x = 440$ ①

$\log_2 x = \frac{\cancel{440}^{40} \times 2}{10 \times \cancel{11}}$

$\log_2 x = 8$

$\therefore x = 2^8$ or $x = 256$ ①

b i) from standard table of integration,

$f'(x) = \frac{1}{\sqrt{x^2+1}}$ ①

ii) $\int_0^1 \frac{dx}{\sqrt{x^2+1}} = \left[\ln\left(x + \sqrt{x^2+1}\right)\right]_0^1 = \ln(1+\sqrt{2}) - \ln 1$ ②

$= \ln(1+\sqrt{2})$ #

c) i) $y = \frac{x^4}{16}$

$x^2 = \sqrt{16y} = 4\sqrt{y}$

$V = \int_0^h \pi x^2 \, dy = \pi \int_0^h 4\sqrt{y} \, dy$ (1)

$V = \frac{2}{3} \cdot 4\pi \left[y^{3/2} \right]_0^h dy$ (1)

$V = \frac{8}{3}\pi h^{3/2}$ cm^3 #

ii) $\frac{dV}{dt} \propto \sqrt{h}$

$\frac{dV}{dt} = k\sqrt{h}$ k is a constant ($\frac{1}{2}$)

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$V = \frac{8}{3}\pi h^{3/2}$

$\frac{dV}{dh} = \frac{8\pi}{\cancel{3}} \cdot \frac{\cancel{3}}{2} h^{\frac{1}{2}} = 4\pi h^{\frac{1}{2}}$ (1)

$\therefore \frac{dh}{dt} = \frac{1}{4\pi \cancel{h^{\frac{1}{2}}}} \cdot k \cancel{\sqrt{h}}$

$\frac{dh}{dt} = \frac{k}{4\pi}$ which is a constant ($\frac{1}{2}$)

$\therefore$ the water level in the tank falls at a constant rate.

Q3 4

a) $y = e^{-x} \sin x \qquad x > 0$

$y' = e^{-x}\cos x - e^{-x}\sin x$ (1)

$y' = e^{-x}(\cos x - \sin x) = 0$ when $\cos x = \sin x \qquad (e^{-x} \neq 0)$

$\therefore y' = 0$ when $\tan x = 1$ (1)

For 1^{st} stationary pt. $x = \frac{\pi}{4}$,

$y = e^{-\frac{\pi}{4}} \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}}$

$y'' = e^{-x}[-\sin x - \cos x] - e^{-x}(\cos x - \sin x)$

$= \cancel{-e^{-x}\sin x} - e^{-x}\cos x - e^{-x}\cos x + \cancel{e^{-x}\sin x}$ (1)

$= -2e^{-x}\cos x$

when $x = \frac{\pi}{4}$ $\qquad y'' = -2e^{-\frac{\pi}{4}}\cos\frac{\pi}{4} = -2e^{-\frac{\pi}{4}} \cdot \frac{1}{\sqrt{2}} = -\sqrt{2}e^{-\frac{\pi}{4}} < 0$

$\therefore$ relative max at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}}\right)$ (1)

b i) when $n=1$, $(1+p) - 1 = p$ (1)

$p \div p = 1 \quad \therefore$ true

Assume true for $n = k$

$(1+p)^k - 1 = p \cdot m$ where m is a positive integer ≥ 1 (1)

ie $(1+p)^k = 1 + p \cdot m$

We want to prove $(1+p)^{k+1} - 1$ is divisible by p:

$(1+p)^{k+1} - 1 = (1+p)^k(1+p) - 1$ (2)

$= (1+pm)(1+p) - 1$

$= \cancel{1} + pm + p + p^2 m \cancel{-1}$

$= p(m + 1 + p \cdot m)$

Since P, m are both positive integers

$(m + 1 + pm)$ is also a positive integer ≥ 1

i.e. $(1+p)^{k+1} - 1$ is divisible by p

By the principle of Mathematical Induction,

$(1+p)^n - 1$ is divisible by p for all positive integers $n \geq 1$

ii) $12^n - 4^n - 3^n + 1 = (4^n - 1)(3^n - 1)$

$= \left[(1+3)^n - 1\right]\left[(1+2)^n - 1\right]$ ①

From part i $(1+3)^n - 1$ is divisible by 3 i.e. $(1+3)^n - 1 = 3Q$

$(1+2)^n - 1$ is divisible by 2 i.e. $(1+2)^n - 1 = 3P$

where P, Q are both positive integers ≥ 1

$\therefore 12^n - 4^n - 3^n + 1 = (3Q)(2P)$ ①

$= 6PQ$ i.e. divisible by 6

(Since P, Q are both positive integers ≥ 1
PQ is also a positive integer ≥ 1)

Q 4

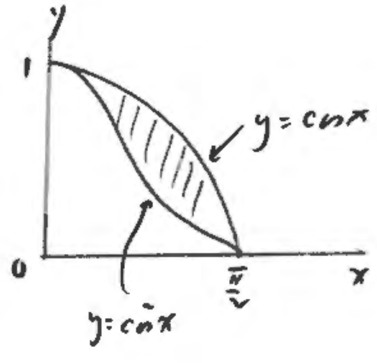
a i) $A = \int_0^{\frac{\pi}{2}} \cos x - \cos^2 x \, dx$

$= \left[\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} \, dx$ ①

$= 1 - 0 - \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}}$

$= 1 - \left[\frac{\pi}{4} + 0 - 0\right]$ ①

$= \frac{4 - \pi}{4}$ sq. units



27)

$$V = \pi \int_0^{\frac{\pi}{2}} \cos^2 x - (\cos^2 x)^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \cos^2 x - \cos^4 x \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left[\frac{1 + \cos 2x}{2} - \left(\frac{1 + (\cos 2x)}{2} \right)^2 \right] dx \qquad (1)$$

$$= \pi \left[\left(\frac{x}{2} + \frac{\sin 2x}{4} \right) \right]_0^{\frac{\pi}{2}} - \frac{\pi}{4} \int_0^{\frac{\pi}{2}} 1 + 2\cos 2x + (\cos 2x)^2 \, dx$$

$$= \pi \left(\frac{\pi}{4} \right) - \frac{\pi}{4} \left[\left(x + \frac{2 \sin 2x}{2} \right) \right]_0^{\frac{\pi}{2}} - \frac{\pi}{4} \int_0^{\frac{\pi}{2}} (\cos 2x)^2 dx$$

$$= \frac{\pi^2}{4} - \frac{\pi}{4} \left(\frac{\pi}{2} \right) - \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{1 + \cos 4x}{2} \, dx \qquad (1)$$

$$= \frac{\pi^2}{8} - \frac{\pi}{8} \left[x + \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{8} - \frac{\pi}{8} \left[\frac{\pi}{2} + 0 \right]$$

$$= \frac{\pi^2}{8} - \frac{\pi^2}{16}$$

$$= \frac{\pi^2}{16} \text{ cubic units} \qquad (1)$$

#

4 b) i) 1 revolution is 2π

4 revolutions is 8π

$\therefore \frac{d\theta}{dt} = 8\pi$ (1)

#

ii) $\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$

$\tan\theta = \frac{x}{3}$

$x = 3\tan\theta$

$\frac{dx}{d\theta} = 3\sec^2\theta$

$\frac{dx}{dt} = 3\sec^2\theta \cdot 8\pi$ (1)

At P, $\theta = 0$ $\sec^2 0 = 1$

$\frac{dx}{dt} = 3 \cdot 1 \cdot 8\pi = 24\pi$ km/min (1)

#

iii) When $x = 2$

$\cos\theta = \frac{3}{\sqrt{13}}$ (1)

$\frac{dx}{dt} = 24\pi\sec^2\theta$

$= \frac{24\pi}{\left(\frac{3}{\sqrt{13}}\right)^2}$

$= \frac{\cancel{24}^{8}\pi}{\cancel{9}_{3}} \times 13$

$= \frac{104\pi}{3}$ km/min (1)

#

$2\ln x - \frac{1}{2}\ln(1-x^2)$

$$\frac{dy}{dx} = \frac{2}{x} - \frac{1}{2}\cdot\frac{-2x}{1-x^2}$$

$$= \frac{2}{x} + \frac{x}{1-x^2}$$

#

) gradient of $FG = \tan(180^\circ - \theta) = -\tan\theta$

$y - 3 = -\tan\theta\,(x-1)$

$y - 3 = \tan\theta - (\tan\theta)x$

$y = -(\tan\theta)x + \tan\theta + 3$

area $\Delta OFG = \frac{1}{2}\times(FO)\times(GO)$

$x = 0 \quad y = \tan\theta + 3 \qquad \therefore FO = \tan\theta + 3$

$y = 0 \quad x = \dfrac{\tan\theta + 3}{\tan\theta} \quad \therefore GO = \dfrac{\tan\theta + 3}{\tan\theta}$

$$\Delta OFG = \frac{1}{2}\left(\tan\theta + 3\right)\left(\frac{\tan\theta + 3}{\tan\theta}\right)$$

$$= \frac{1}{2}\frac{(\tan\theta + 3)^2}{\tan\theta}$$

(1)

$$A = \frac{(\tan\theta)^2 + 6\tan\theta + 9}{2\tan\theta}$$

$$A = \frac{\tan\theta}{2} + 6 + \frac{9}{2}\cot\theta$$

$$A' = \frac{1}{2}\sec^2\theta - \frac{9}{2}\operatorname{cosec}^2\theta \qquad (1)$$

$$A' = \frac{1}{2\cos^2\theta} - \frac{9}{2\sin^2\theta}$$

$A' = 0$ when $\frac{1}{\cancel{2}\cos^2\theta} = \frac{9}{\cancel{2}\sin^2\theta}$

$\therefore \tan^2\theta = 9$

$\tan\theta = 3, -3$

But $0 < \theta < \frac{\pi}{2}$ $\therefore \tan\theta = 3$ (1)

$\theta = 71^\circ 33' 54''$

θ	71°	$71^\circ 33' 54''$	72°
A'	-0.32	0	0.26

Since A, A' are continuous for $0 < \theta < \frac{\pi}{2}$ and A' changes sign (1)

$\therefore$ rel min at $\theta = 71^\circ 33' 54''$

Since A has only 1 turning point, rel. min is also the absolute min

$\therefore$ Area A is a min at $\theta = 71^\circ 33' 54''$ (1)