Question 1 (9 Marks) Marks

(a) Express $\log\left(\frac{y^3}{x\sqrt{x}}\right)$ in terms of p and q, if $p = \log x$ and $q = \log y$. **2**

(b) Find $\int \frac{dx}{x+\sqrt{x}}$ using the substitution $u = \sqrt{x}$. **3**

(c) (i) Differentiate xe^{4x}. **1**

(ii) Hence, or otherwise, evaluate $\int_0^1 xe^{4x}\, dx$. **3**

Question 2 (9 Marks) START A NEW PAGE Marks

(a) Find the values of k, if $y = \cos kx$ is a solution of the equation $\frac{d^2y}{dx^2} + 4y = 0$. **2**

(b) Find acute values for α so that the infinite series **3**

$$1 - 4\sin 2\alpha + 16\sin^2 2\alpha - 64\sin^3 2\alpha + \ldots$$

will have a limiting sum equal to $\frac{1}{3}$.

(c) (i) Express $\frac{2x+3}{x+2}$ in the form $A + \frac{B}{x+2}$ where A and B are rational numbers. **1**

(ii) Find the area bounded by the curve $y = \frac{2x+3}{x+2}$ and the co-ordinate axes. **3**

Express your answer in the form $P + Q\ln 2$ where P and Q are rational numbers.

Question 3 (9 Marks) START A NEW PAGE Marks

(a) If $f'(x) = 4\sec^2 3x$, find $f(x)$ given that $f\left(\frac{\pi}{4}\right) = \frac{2}{3}$. **2**

(b) Solve $2^{x+1} = 3^{x-1}$ giving your answer correct to 2 decimal places. **3**

(c) Prove by the Principle of Mathematical Induction that **4**

$2\times 1! + 5\times 2! + 10\times 3! + \ldots + (n^2+1)n! = n(n+1)!$ for $n = 1, 2, 3, \ldots$

Question 4 (9 Marks) START A NEW PAGE Marks

(a) Find the equation of the tangent to the curve $y = x^2 \ln x$ at the point where $x = e$. **3**

(b) A radio news item stated that "the incidence of hip fractures in older adults has fallen by 20% over the last 10 years". Assuming that the rate of decline was compounded annually, how many more years will be required for the incidence of hip fractures to reduce to 50% if the above rate of decline remains unchanged? Give your answer correct to the nearest year. **3**

(c) A sphere is contracting so that its volume is decreasing at a constant rate of $20\ mm^3/\sec$. Find the rate of change of the surface area when the radius of the sphere is 5 *mm*. **3**

Question 5 (9 Marks) START A NEW PAGE Marks

(a) Sam buys a car valued at $6000 and agrees to pay an initial deposit of 20% and the remaining balance in several equal repayments over 5 years. In his loan agreement Sam is to make equal repayments at the end of each quarter (i.e. at the end of every 3 months). The interest, which is at a rate of 6% pa, is to be compounded on the outstanding balance of the loan at the end of each month and is added to the outstanding balance prior to any repayments.

(i) Find the amount owed before to the first repayment is due. **1**

(ii) Find the value of each repayment. **3**

(iii) Find the total amount that Sam pays for the car. **1**

(b) Given that $A_n = \sum_{k=0}^{k=2n} 5^{3k-2n}$.

(i) Evaluate A_1 and A_2. **2**

(ii) Find a simplified formula for A_n that does not involve sigma notation. **2**

Question 6 (9 Marks) START A NEW PAGE Marks

(a) (i) Differentiate $\ln(\sec 2x + \tan 2x)$. **1**

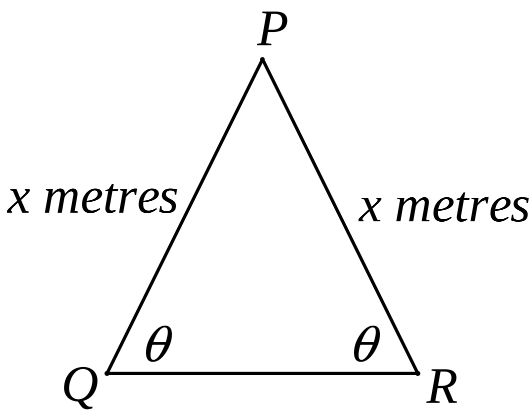
(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{6}} \sec 2x\, dx$. **2**

(b) (i) Find the x-coordinate of the point of intersection of the curves $y = 3\cos x$ and $y = 8\tan x$ for $0 \le x \le \frac{\pi}{2}$. **2**

(ii) Find the volume of the solid formed when the area bounded by the curves $y = 3\cos x$, $y = 8\tan x$ and the x-axis for $0 \le x \le \frac{\pi}{2}$ is rotated one revolution about the x-axis. Give your answer correct to 1 decimal place. **4**

Question 7 (9 Marks) START A NEW PAGE Marks

ΔPQR has a perimeter of 4 metres.
Let $PQ = PR = x$ metres and $\angle PQR = \angle PRQ = \theta$.
(see diagram)



(i) Show that $QR = 2x\cos\theta$. **1**

(ii) Show that the area, $A\ m^2$, of ΔPQR is given by: **2**

$$A = \frac{2\sin 2\theta}{(1+\cos\theta)^2}.$$

(iii) Show that $\dfrac{dA}{d\theta} = \dfrac{4(\cos 2\theta + \cos\theta)}{(1+\cos\theta)^3}$. **3**

(iv) Hence find the value of θ so that the triangle has its greatest area. **3**

THIS IS THE END OF THE EXAMINATION PAPER

Question 1 (9 Marks) Marks

(a) Express $\log\left(\dfrac{y^3}{x\sqrt{x}}\right)$ in terms of p and q, if $p = \log x$ and $q = \log y$. **2**

Solution:

$$\log\left(\frac{y^3}{x\sqrt{x}}\right) = 3\log y - 1\tfrac{1}{2}\log x$$
$$= 3q - 1\tfrac{1}{2}p$$

(b) Find $\displaystyle\int \frac{dx}{x+\sqrt{x}}$ using the substitution $u = \sqrt{x}$. **3**

Solution:

$$u = \sqrt{x} = x^{\frac{1}{2}}$$
$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$
$$du = \frac{dx}{2\sqrt{x}}$$
$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2\sqrt{x}}{x+\sqrt{x}}.\frac{dx}{2\sqrt{x}}$$
$$= \int \frac{2u}{u^2+u}.\,du$$
$$= \int \frac{2}{u+1}.\,du$$
$$= 2\ln(u+1)+c$$
$$= 2\ln(\sqrt{x}+1)+c$$

(c) (i) Differentiate xe^{4x}. **1**

Solution:

$$\frac{d}{dx}\left(xe^{4x}\right) = (1)\left(e^{4x}\right) + (x)\left(4e^{4x}\right)$$
$$= e^{4x} + 4xe^{4x}$$

(ii) Hence, or otherwise, evaluate $\int_0^1 xe^{4x}\ dx$. **3**

Solution:

$$\frac{d}{dx}\left(xe^{4x}\right) = e^{4x} + 4xe^{4x}$$

$$4xe^{4x} = \frac{d}{dx}\left(xe^{4x}\right) - e^{4x}$$

$$xe^{4x} = \frac{1}{4}\left\{\frac{d}{dx}\left(xe^{4x}\right) - e^{4x}\right\}$$

$$\int_0^1 xe^{4x}\ dx = \frac{1}{4}\left\{\int_0^1 \left[\frac{d}{dx}\left(xe^{4x}\right) - e^{4x}\right] dx\right\}$$

$$= \frac{1}{4}\left[xe^{4x} - \frac{1}{4}e^{4x}\right]_0^1$$

$$= \frac{1}{4}\left\{\left(e^4 - \frac{1}{4}e^4\right) - \left(0 - \frac{1}{4}e^0\right)\right\}$$

$$= \frac{1}{4}\left\{\left(\frac{3}{4}e^4\right) - \left(-\frac{1}{4}\right)\right\}$$

$$= \frac{1}{16}\left(3e^4 + 1\right)$$

Question 2 (9 Marks) START A NEW PAGE Marks

(a) Find the values of k, if $y = \cos kx$ is a solution of the equation $\dfrac{d^2y}{dx^2} + 4y = 0$. **2**

Solution:

$$\frac{dy}{dx} = -k \sin kx$$

$$\frac{d^2y}{dx^2} = -k^2 \cos kx$$

$$\frac{d^2y}{dx^2} + 4y = 0 \Rightarrow -k^2 \cos kx + 4\cos kx = 0$$

$$\left(-k^2 + 4\right)\cos kx = 0$$

$$k^2 = 4$$

$$k = \pm 2$$

(b) Find acute values for α so that the infinite series **3**

$$1 - 4\sin 2\alpha + 16\sin^2 2\alpha - 64\sin^3 2\alpha + \ldots$$

will have a limiting sum equal to $\dfrac{1}{3}$.

Solution:

$$a = 1,\ r = -4\sin 2\alpha$$

$$S = \frac{a}{1-r}$$

$$\frac{1}{3} = \frac{1}{1 + 4\sin 2\alpha}$$

$$1 + 4\sin 2\alpha = 3$$

$$4\sin 2\alpha = 2$$

$$\sin 2\alpha = \frac{1}{2}$$

$$2\alpha = \frac{\pi}{6}\ or\ \frac{5\pi}{6}$$

$$\alpha = \frac{\pi}{12}\ or\ \frac{5\pi}{12}$$

(c) (i) Express $\dfrac{2x+3}{x+2}$ in the form $A + \dfrac{B}{x+2}$ where A and B are rational numbers. **1**

Solution:

$$\frac{2x+3}{x+2} = \frac{2x+4-1}{x+2}$$

$$= \frac{2(x+2)-1}{x+2}$$

$$= 2 - \frac{1}{x+2}$$

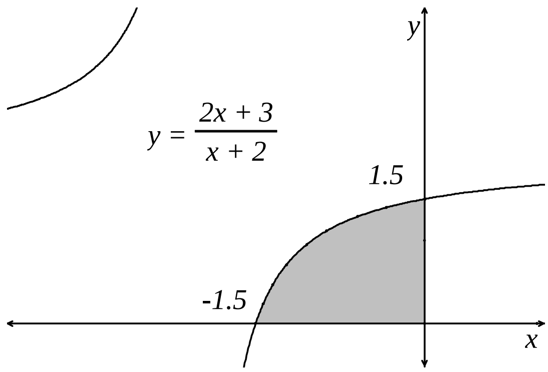
(ii) Find the area bounded by the curve $y = \dfrac{2x+3}{x+2}$ and the co-ordinate axes. **3**

Express your answer in the form $P + Q\ln 2$ where P and Q are rational numbers.

Solution:

$$A = \int_{-1.5}^{0} \left(2 - \frac{1}{x+2}\right) dx$$
$$= \left[2x - \ln(x+2)\right]_{-1.5}^{0}$$
$$= (0 - \ln 2) - \left(-3 - \ln\left(\frac{1}{2}\right)\right)$$
$$= -\ln 2 + 3 + \ln\left(\frac{1}{2}\right)$$
$$= -\ln 2 + 3 - \ln 2$$
$$= 3 - 2\ln 2$$

area $= (3 - 2\ln 2)\, u^2$



Question 3 (9 Marks) START A NEW PAGE Marks

(a) If $f'(x) = 4\sec^2 3x$, find $f(x)$ given that $f\left(\frac{\pi}{4}\right) = \frac{2}{3}$. **2**

Solution:

$f'(x) = 4\sec^2 3x$

$f(x) = \frac{4}{3}\tan 3x + c$

given $f\left(\frac{\pi}{4}\right) = \frac{2}{3}$

$\frac{2}{3} = \frac{4}{3}\tan\frac{3\pi}{4} + c$

$\frac{2}{3} = -\frac{4}{3} + c$

$c = 2$

$\therefore\ f(x) = \frac{4}{3}\tan 3x + 2$

(b) Solve $2^{x+1} = 3^{x-1}$ giving your answer correct to 2 decimal places. **3**

Solution:

$2^{x+1} = 3^{x-1}$

$\log\left(2^{x+1}\right) = \log\left(3^{x-1}\right)$

$(x+1)\log 2 = (x-1)\log 3$

$x\log 2 + \log 2 = x\log 3 - \log 3$

$x\log 3 - x\log 2 = \log 3 + \log 2$

$x(\log 1.5) = \log 6$

$x = \frac{\log 6}{\log 1.5}$

$x = 4.42$ (to 2 d.p.)

(c) Prove by the Principle of Mathematical Induction that **4**
$2\times 1! + 5\times 2! + 10\times 3! + \ldots + (n^2+1)n! = n(n+1)!$ for $n = 1, 2, 3, \ldots$

Solution:

when $n = 1$

$LHS = 2\times 1!$

$\quad\;\; = 2$

$RHS = 1(1+1)!$

$\quad\;\; = 2$

$\therefore LHS = RHS$

$\therefore$ true for $n = 1$

Assume true for $n = k \quad (k \in Z^+)$

i.e. $\quad 2\times1! + 5\times2! + 10\times3! + \ldots + (k^2+1)k! = k(k+1)!$

Prove true for $n = k+1$

i.e. $\quad 2\times1! + 5\times2! + 10\times3! + \ldots + (k^2+1)k! + \left[(k+1)^2+1\right](k+1)! = (k+1)(k+2)!$

$$
\begin{aligned}
\text{Now LHS} &= 2\times1! + 5\times2! + 10\times3! + \ldots + (k^2+1)k! + \left[(k+1)^2+1\right](k+1)! \\
&= k(k+1)! + \left[(k+1)^2+1\right](k+1)! \quad (\text{by assumption}) \\
&= (k+1)!\left\{k + k^2 + 2k + 2\right\} \\
&= (k+1)!\left\{k^2 + 3k + 2\right\} \\
&= (k+1)!(k+1)(k+2) \\
&= (k+1)(k+2)! \\
&= RHS
\end{aligned}
$$

$\therefore$ true by the Principle of Mathematical Induction

Question 4 (9 Marks) START A NEW PAGE Marks

(a) Find the equation of the tangent to the curve $y = x^2 \ln x$ at the point where $x = e$. **3**

Solution:

$$y = x^2 \ln x$$

$$\frac{dy}{dx} = (2x)(\ln x) + (x^2)\left(\frac{1}{x}\right)$$

$$= 2x \ln x + x$$

$$\text{when } x = e,\ \frac{dy}{dx} = 2e \ln e + e$$

$$= 3e$$

$$\text{when } x = e,\ y = e^2 \ln e$$

$$= e^2$$

$$\text{tangent is : } \quad y - e^2 = 3e(x - e)$$

$$y = 3ex - 2e^2$$

(b) A radio news item stated that "the incidence of hip fractures in older adults has fallen by 20% over the last 10 years". Assuming that the rate of decline was compounded annually, how many <u>more</u> years will be required for the incidence of hip fractures to reduce to 50% if the above rate of decline remains unchanged? Give your answer correct to the nearest year. **3**

Solution:

$$A = P(1-r)^n$$

$$\text{when } A = 0.8P$$

$$0.8P = P(1-r)^{10}$$

$$(1-r)^{10} = 0.8$$

$$1 - r = \sqrt[10]{0.8}$$

$$r = 1 - \sqrt[10]{0.8}$$

$$\text{when } A = 0.5P$$

$$0.5P = P(1-r)^n \quad \text{where } r = 1 - \sqrt[10]{0.8}$$

$$(1-r)^n = 0.5$$

$$n \ln(1-r) = \ln(0.5)$$

$$n = \frac{\ln(0.5)}{\ln\left(1 - \left(1 - \sqrt[10]{0.8}\right)\right)}$$

$$= \frac{\ln(0.5)}{\ln\left(\sqrt[10]{0.8}\right)}$$

$$= 31.06 \quad (\text{to } 2\ d.p.)$$

$$\text{extra time} = 21 \text{ years } (\text{to nearest year})$$

(c) A sphere is contracting so that its volume is decreasing at a constant rate of $20\ mm^3/\sec$. **3**
Find the rate of change of the surface area when the radius of the sphere is 5 *mm*.

Solution:

$$\frac{dA}{dt}=\frac{dA}{dR}\times\frac{dR}{dt}$$
$$=\frac{dA}{dR}\times\frac{dR}{dV}\times\frac{dV}{dt} \quad \text{since } \frac{dr}{dt}=\frac{dR}{dV}\times\frac{dV}{dt}$$

$$V=\frac{4}{3}\pi R^3$$
$$\frac{dV}{dR}=4\pi R^2$$

$$A=4\pi R^2$$
$$\frac{dA}{dR}=8\pi R$$

$$\frac{dA}{dt}=\frac{dA}{dR}\times\frac{dR}{dV}\times\frac{dV}{dt}$$
$$=(8\pi R)\times\left(\frac{1}{4\pi R^2}\right)\times(-20)$$
$$=-\frac{40}{R}$$

When $R=5$

$$\frac{dA}{dt}=-\frac{40}{5}$$
$$=-8$$

Change is $-8\ mm^2/\sec$

Question 5 (9 Marks) START A NEW PAGE Marks

(a) Sam buys a car valued at \$6000 and agrees to pay an initial deposit of 20% and the remaining balance in several equal repayments over 5 years. In his loan agreement Sam is to make equal repayments at the end of each quarter (i.e. at the end of every 3 months). The interest, which is at a rate of 6% pa, is to be compounded on the outstanding balance of the loan at the end of each month and is added to the outstanding balance prior to any repayments.

(i) Find the amount owed before to the first repayment is due. **1**

Solution:

$$\begin{aligned}\text{Amount owing after deposit} &= \$6000\times 0.8\\ &= \$4800\\ \text{Amount to repay after 1st interest} &= \$4800\times 1.005^3\\ &= \$4872.36 \quad (\text{to nearest cent})\end{aligned}$$

(ii) Find the value of each repayment. **3**

Solution:

Let $\$A_n$ be the amount owing after the n^{th} repayment and $\$P$ be the value of a repayment.

$$\begin{aligned}A_1 &= 4800\times 1.005^3 - P\\ A_2 &= A_1\times 1.005^3 - P\\ &= \left(4800\times 1.005^3 - P\right)\times 1.005^3 - P\\ &= 4800\times 1.005^6 - 1.005^3 P - P\\ &= 4800\times 1.005^6 - \left(1.005^3 + 1\right)P\end{aligned}$$

$$A_{20} = 4800\times 1.005^{60} - \left(1.005^{57} + 1.005^{54} + \ldots + 1.005^3 + 1\right)P$$

When loan repaid $A_{20} = 0$

$$\therefore\quad 4800\times 1.005^{60} - \left(1.005^{57} + 1.005^{54} + \ldots + +1.005^3 + 1\right)P = 0$$

$$\left(1.005^{57} + 1.005^{54} + \ldots + +1.005^3 + 1\right)P = 4800\times 1.005^{60}$$

$$\frac{1\left[\left(1.005^3\right)^{20} - 1\right]}{1.005^3 - 1}P = 4800\times 1.005^{60}$$

$$\begin{aligned}P &= \frac{4800\times 1.005^{60}}{\dfrac{\left[1.005^{60} - 1\right]}{1.005^3 - 1}}\\ &= 279.79 \quad (\text{to } 2\ d.p.)\end{aligned}$$

Repayment value = \$279.27 (to nearest cent)

(iii) Find the total amount that Sam pays for the car. **1**

Solution:

$$\text{Total paid} = \$6000 \times 0.2 + 279.79 \times 20$$
$$= \$6795.80$$

(b) Given that $A_n = \sum_{k=0}^{k=2n} 5^{3k-2n}$.

(i) Evaluate A_1 and A_2 . **2**

Solution:

$$A_1 = \sum_{k=0}^{k=2} 5^{3k-2}$$
$$= 5^{-2} + 5^{-1} + 5^{4}$$
$$= 630.04$$

$$A_2 = \sum_{k=0}^{k=4} 5^{3k-4}$$
$$= 5^{-4} + 5^{-1} + 5^{2} + 5^{5} + 5^{8}$$
$$= 393775.2016$$

(ii) Find a simplified formula for A_n that does not involve sigma notation. **2**

Solution:

$$A_n = 5^{-2n} + 5^{3-2n} + 5^{6-2n} + \ldots + 5^{4n} \ldots\ldots(1)$$
$$r = 5^{3}$$
$$5^{3} A_n = 5^{3-2n} + 5^{6-2n} + 5^{9-2n} + \ldots + 5^{4n} + 5^{3+4n} \ldots\ldots(2)$$
$$(2)-(1)$$
$$124 A_n = 5^{3+4n} - 5^{-2n}$$
$$A_n = \frac{5^{4n+3} - 5^{-2n}}{124}$$

Question 6 (9 Marks) START A NEW PAGE Marks

(a) (i) Differentiate $\ln(\sec 2x + \tan 2x)$. **1**

Solution:

$$\begin{aligned}\frac{d}{dx}\{\ln(\sec 2x+\tan 2x)\} &= \frac{2\sec x\tan 2x + 2\sec^2 2x}{\sec 2x+\tan 2x}\\ &= \frac{2\sec x(\tan 2x+\sec 2x)}{\sec 2x+\tan 2x}\\ &= 2\sec 2x\end{aligned}$$

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{6}} \sec 2x\, dx$. **2**

Solution:

$$\sec 2x = \frac{1}{2}\frac{d}{dx}\{\ln(\sec 2x+\tan 2x)\}$$

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \sec 2x\, dx &= \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{d}{dx}\{\ln(\sec 2x+\tan 2x)\}dx\\ &= \frac{1}{2}[\ln(\sec 2x+\tan 2x)]_0^{\frac{\pi}{6}}\\ &= \frac{1}{2}\left\{\left(\ln\left(\sec\frac{\pi}{3}+\tan\frac{\pi}{3}\right)-\ln(\sec 0+\tan 0)\right)\right\}\\ &= \frac{1}{2}\{\ln(2+\sqrt{3})-\ln(1)\}\\ &= \frac{1}{2}\ln(2+\sqrt{3})\end{aligned}$$

(b) (i) Find the x-coordinate of the point of intersection of the curves $y = 3\cos x$ and $y = 8\tan x$ for $0 \le x \le \frac{\pi}{2}$. **2**

Solution:

$$8\tan x = 3\cos x$$

$$8\frac{\sin x}{\cos x} = 3\cos x$$

$$8\sin x = 3\cos^2 x$$

$$8\sin x = 3(1-\sin^2 x)$$

$$3\sin^2 x + 8\sin x - 3 = 0$$

$$(3\sin x - 1)(\sin x + 3) = 0$$

$$\sin x = \frac{1}{3} \quad (\sin x \ne -3)$$

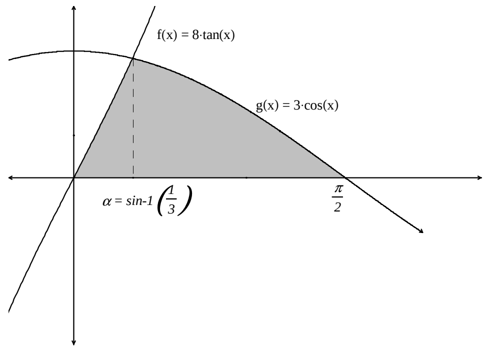
$$x = \sin^{-1}\left(\frac{1}{3}\right) \quad (x \text{ is acute})$$

(ii) Find the volume of the solid formed when the area bounded by the curves $y = 3\cos x$, $y = 8\tan x$ and the x-axis for $0 \le x \le \dfrac{\pi}{2}$ is rotated one revolution about the x-axis. Give your answer correct to 1 decimal place. **4**

Solution:

$$V = \pi\int_0^{\alpha} (8\tan x)^2\, dx + \pi\int_0^{\alpha} (3\cos x)^2\, dx$$

where $\alpha = \sin^{-1}\left(\dfrac{1}{3}\right)$



$$V = 64\pi\int_0^{\alpha} \tan^2 x\, dx + 9\pi\int_{\alpha}^{\frac{\pi}{2}} \cos^2 x\, dx$$

$$= 64\pi\int_0^{\alpha} \left(\sec^2 x - 1\right) dx + 9\pi\int_{\alpha}^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\, dx$$

$$= 64\pi\left[\tan x - x\right]_0^{\alpha} + \frac{9\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_{\alpha}^{\frac{\pi}{2}}$$

$$= 64\pi\left\{(\tan\alpha - \alpha) - (0)\right\} + 4.5\pi\left\{\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(\alpha - \frac{1}{2}\sin 2\alpha\right)\right\}$$

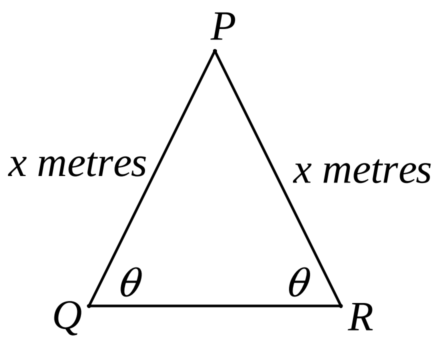
$$= 64\pi\left\{\tan\alpha - \alpha\right\} + 4.5\pi\left\{\frac{\pi}{2} - \alpha + \frac{1}{2}\sin 2\alpha\right\}$$

$$\approx 15.7172.....$$

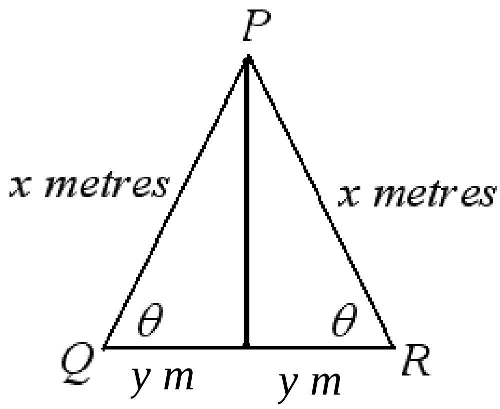
Volume $= 15.7\ u^3$ (to 1 d.p.)

Question 7 (9 Marks) START A NEW PAGE Marks

ΔPQR has a perimeter of 4 metres.
Let $PQ = PR = x$ metres and $\angle PQR = \angle PRQ = \theta$.
(see diagram)



(i) Show that $QR = 2x\cos\theta$. **1**



Solution:
Altitude of isosceles triangle is bisector of base

$$\frac{y}{x} = \cos\theta$$
$$y = x\cos\theta$$
$$QR = 2y$$
$$= 2x\cos\theta$$

- could also use cosine rule

$$QR^2 = x^2 + x^2 - 2.x.x.\cos(\pi - 2\theta)$$
$$= 2x^2 + 2x^2\cos 2\theta$$
$$= 2x^2(1 + 2\cos 2\theta)$$
$$= 2x^2(2\cos^2\theta)$$
$$= 4x^2\cos^2\theta$$
$$QR = 2x\cos\theta \quad (QR > 0, x \text{ acute})$$

(ii) Show that the area, $A\ m^2$, of ΔPQR is given by: **2**

$$A = \frac{2\sin 2\theta}{(1+\cos\theta)^2}.$$

Solution:

$$Area = \frac{1}{2}PQ.QR.\sin\theta$$
$$= \frac{1}{2}x.2x.\cos\theta.\sin\theta$$
$$= \frac{1}{2}x^2\sin 2\theta$$

But perimeter is 4 m

$\therefore 2x + 2x\cos\theta = 4$

$x(1+\cos\theta) = 2$

$x = \dfrac{2}{1+\cos\theta}$

$A = \dfrac{1}{2}.\left(\dfrac{2}{1+\cos\theta}\right)^2 \sin 2\theta$

$= \dfrac{2\sin 2\theta}{(1+\cos\theta)^2}$

(iii) Show that $\dfrac{dA}{d\theta} = \dfrac{4(\cos 2\theta + \cos\theta)}{(1+\cos\theta)^3}$. **3**

Solution:

$A = 2\left\{\dfrac{\sin 2\theta}{(1+\cos\theta)^2}\right\}$

$\dfrac{dA}{d\theta} = 2\left\{\dfrac{(1+\cos\theta)^2(2\cos 2\theta) - 2(1+\cos\theta)(-\sin\theta)\sin 2\theta}{(1+\cos\theta)^4}\right\}$

$= 2\left\{\dfrac{(1+\cos\theta)(2\cos 2\theta) + 2\sin\theta\sin 2\theta}{(1+\cos\theta)^3}\right\}$

$= 4\left\{\dfrac{\cos 2\theta + \cos\theta\cos 2\theta + \sin\theta\sin 2\theta}{(1+\cos\theta)^3}\right\}$

$= 4\left\{\dfrac{\cos 2\theta + \cos(2\theta-\theta)}{(1+\cos\theta)^3}\right\}$

$= 4\left\{\dfrac{\cos 2\theta + \cos\theta}{(1+\cos\theta)^3}\right\}$

(iv) Hence find the value of θ so that the triangle has its greatest area. **3**

Solution:

For stat. pt $\dfrac{dA}{d\theta} = 0$

$4\left\{\dfrac{\cos 2\theta + \cos\theta}{(1+\cos\theta)^3}\right\} = 0$

$\cos 2\theta + \cos\theta = 0$

$2\cos^2\theta - 1 + \cos\theta = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \dfrac{1}{2}$ or -1

but θ is acute

$\therefore \theta = 60°$ or $\dfrac{\pi}{3}$

Test nature of stationary point

θ	$59°$	$60°$	$61°$
$\dfrac{dA}{d\theta}$	$\approx$ 0.0524	0	$\approx$ -0.0551
	+	0	-

Gradient changes sign (+ 0 -) therefore stat. pt. is a local max. tp.
Since function is continuous for $0 < \theta < 90°$ and there is only one stat. pt. then the local max. is the absolute max.

$\therefore$ maximum area occurs when $\theta = 60° \quad \left(\text{or } \dfrac{\pi}{3}\right)$.

THIS IS THE END OF THE EXAMINATION PAPER