Name:	
Class:	

YEAR 12

ASSESSMENT TASK 2
TERM 1, 2014

MATHEMATICS
EXTENSION 1

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 2 hours.
- Write in black or blue pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In Question 6 - 12, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 82

Section I: 5 marks

- Attempt Question 1 – 5.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 8 minutes for this section.

Section II: 77 Marks

- Attempt Question 6 - 12
- Answer on blank paper unless otherwise instructed. Start a new page for each new question.
- Allow about 1 hours & 52 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 6, Question 7, etc. Each question must show your Candidate Number.

SECTION I MULTIPLE CHOICE (5 marks)

Attempt Question 1 – 5 (1 mark each)

Allow approximately 8 minutes for this section

1. A telephone number consists of 7 digits. How many different telephone numbers exist if each digit appears only once in the numbers?

(A) $7!$

(B) 10^7

(C) 7^7

(D) $\frac{10!}{3!}$

2. The sum of the first four terms of an arithmetic progression is S_{AP} and the sum of the first four terms of a geometric progression is S_{GP}. If $\frac{S_{AP}}{S_{GP}} = \frac{2}{3}$ and the common difference and common ratio of the arithmetic and geometric progressions are both equal to 2, determine $S_{AP} + S_{GP}$.

(A) 50

(B) 40

(C) 30

(D) 20

3. The domain and range for $y = 3\sin^{-1}2x$ respectively are:

(A) $-3 \le x \le 3$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(B) $-2 \le x \le 2$; $-3\pi \le y \le 3\pi$

(C) $-\frac{1}{2} \le x \le \frac{1}{2}$; $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

(D) $-\frac{1}{3} \le x \le \frac{1}{3}$; $-\frac{2\pi}{3} \le y \le \frac{2\pi}{3}$

P.T.O.

4. Simplify the expression:

$$\frac{2n!}{(n-1)!} + \frac{2(n-1)!\,(n-1)!}{[(n-2)!]^2}$$

(A) $\frac{2n}{n-2}$

(B) $\frac{2n}{(n-1)(n-2)}$

(C) $2n + \frac{2}{(n-2)^2}$

(D) $2(n^2 - n + 1)$

5. If $t = \tan\alpha$, which of the following is an exact expression for $\dfrac{\cos 2\alpha}{1 + \sin 2\alpha}$?

(A) $\frac{t-1}{t+1}$

(B) $\frac{1-t^2}{1+2t}$

(C) $\frac{1-t}{1+t}$

(D) $\frac{1}{t+\sqrt{1+t^2}}$

END OF SECTION I

P.T.O.

SECTION II EXTENDED RESPONSE (77 marks)

Total Marks is 77

Attempt Questions 6 – 12.

Allow approximately 1 hour & 52 minutes for this section.

QUESTION 6 (11 Marks) **COMMENCE A NEW PAGE** **MARKS**

(a) Find $\int \frac{e^{2x}}{1 + e^{2x}}\, dx$ **2**

(b) Evaluate $\int_0^{\pi/3} \cos^2 5x \; dx$. [Leave answer in exact form]. **3**

(c) Let $f(x) = 1 - 2\sin^2 x$

(i) Sketch $y = f(x)$ for the domain $0 \leq x \leq 2\pi$ **1**

(ii) Shade the region on your sketch, bounded by the curve $y = 1 - 2\sin^2 x$, the x-axis and the lines $x = 0$ and $x = \frac{5\pi}{4}$. **1**

(iii) Find the exact value of the area of this shaded region. **2**

(d) Find the sum to n terms of the series **2**

$$9\frac{1}{32} + 11\frac{1}{64} + 13\frac{1}{128} + \ldots$$

P.T.O.

QUESTION 7 (11 Marks) **COMMENCE A NEW PAGE** **MARKS**

(a) How many arrangements of the word 'EQUATION' are there if none of the consonants are together? **2**

(b) (i) Prove that $\frac{d}{dx}[\tan^{-1}(cosec\ x)] = \frac{-\cos x}{\sin^2 x + 1}$, **2**

(ii) Hence prove that

$$\int_{\tan^{-1}(\frac{1}{\sqrt{3}})}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x + 1}\, dx = \tan^{-1}(\tfrac{1}{3}).$$ **2**

(c) The chord PQ of the parabola $x^2 = 4ay$ passes through the focus S. T is a point on the tangent to the parabola at Q such that TP is parallel to the axis of the parabola.

Let the coordinates of P and Q be $(2ap, ap^2)$ and $(2aq, aq^2)$ respectively.

(i) Find the coordinates of T. **3**

(ii) Further, assume that the equation of chord PQ is $y = \frac{1}{2}(p+q)x - apq$.

Hence show that the equation of the locus of T is **2**

$x^2(y + 2a) + 4a^3 = 0$

QUESTION 8 (11 Marks) **COMMENCE A NEW PAGE**

(a) (i) Find $\frac{d}{dx}[\ln x]^n$ **1**

(ii) Hence, find the volume of the solid of revolution formed if the portion of the curve $y = \frac{\ln x}{\sqrt{x}}$ from $x = 1$ to $x = e$ is rotated about the x-axis. **3**

(b) Find the equation of the normal to the curve $y = \ln\left[\frac{1+\sin x}{\cos x}\right]$ at $x = \frac{\pi}{3}$. **4**

Leave answer in the most simplified and exact form as $ax + by + c = 0$.

P.T.O.

(c) The first term of an arithmetic series is $\log_e 2$ and the common difference is $\log_e 4$.

(i) Show that the sum to n terms of this series is $S_n = n^2 \log_e 2$. **1**

(ii) Find the least value of n for which S_n is greater than fifty times the nth term. **2**

QUESTION 9 (11 Marks) COMMENCE A NEW PAGE

(a) If $y = \cos\theta + \sqrt{3}\sin\theta$,

(i) Express y in the form $R\cos(\theta - \alpha)$ where $R > 0$, and $0 \leq \alpha \leq \frac{\pi}{2}$. **2**

(ii) Hence or otherwise, solve $\cos\theta + \sqrt{3}\sin\theta = 1$ for $0 \leq \theta \leq 2\pi$. **2**

(b) Rita borrows \$20 000 from City Credit at 12 % p.a. interest, compounded monthly. She pays it back at regular monthly intervals over 4 years. However, because she is a good customer she is given the first two months interest free.

(i) If each monthly payment is M, find in terms of M:

(α) the amount she owes after the second payment. **1**

(β) the amount she owes after the fourth payment. **1**

(ii) Show that the amount of each monthly payment can be expressed as

$$M = \frac{200\,(1.01)^{46}}{1.02(1.01)^{46} - 1}$$ **3**

(iii) At the end of the second year Rita decides to use her bonus to make a once off payment of \$2 000 into her loan account. Assuming she then continues to make the same arranged monthly payment found in part (ii), how soon from this once off payment, will she be able to pay off her loan? **2**

P.T.O.

QUESTION 10 (11 Marks) **COMMENCE A NEW PAGE** **MARKS**

(a) Let $y = (x^2 - 3)e^{-x}$

(i) Show that the stationary points occur at $x = -1$ and $x = 3$ and determine their nature. **3**

4

(ii) Make a neat sketch, including the intercepts with the axes, coordinates of any turning points and any asymptotes. **4**

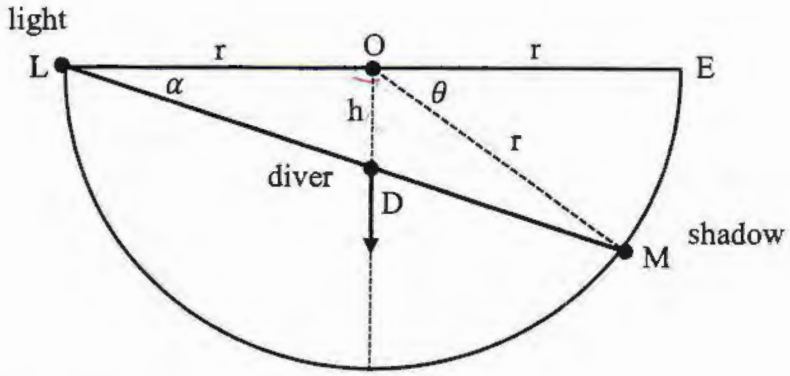
Express all intercepts, turning points and asymptotes as exact values.

(iii) Using the sketch drawn and another sketch, solve the inequality $(e^x - e^{-x})(x^2 - 3) \geq 0$. Show clearly how you obtained your answer. **2**

(b) If $y = \sin x$, prove $\frac{dy}{dx} = \cos x$. **2**

QUESTION 11 (11 Marks) **COMMENCE A NEW PAGE**

(a) A diver is descending vertically from the centre of a hemispherical tank of radius r metres, at the rate of 2 metres per second. A light, L, at the edge of the tank throws the diver's shadow, M, on the curved surface of the tank, as shown in the diagram below. The diver at D is h metres from O at time $t = 0$.



(i) Verify that $\alpha = \tan^{-1}\frac{h}{r} = \frac{\theta}{2}$ **3**

(ii) If s is the distance the shadow moves through along the edge of the curve, how fast is the shadow moving along the tank's edge when the diver is halfway down i.e. find $\frac{ds}{dt}$. **4**

P.T.O.

(b) Prove by mathematical induction that $4^n \geq 1 + 3n$ for $n = 1, 2, 3, \ldots$ **4**

QUESTION 12 (11 Marks) **COMMENCE A NEW PAGE** **MARKS**

(a) Evaluate $\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\, dx$, using the substitution $x = \sec\theta$. **4**

(b) (i) Write down an expression for $\tan(A - B)$ in terms of $\tan A$ and $\tan B$. **1**

(ii) Deduce that for positive integer values of n **2**

$$1 + \tan n\theta \tan(n+1)\theta = \frac{\tan(n+1)\theta - \tan n\theta}{\tan\theta}$$

(iii) ~~(iv)~~ Hence prove that the sum to n terms of the series **4**

$$\tan\theta\tan 2\theta + \tan 2\theta \tan 3\theta + \tan 3\theta \tan 4\theta + \ldots\ldots$$

is

$$S_n = \cot\theta\tan(n+1)\theta - (n+1)$$

☺ **END OF EXAMINATION** ☺

JRAHS : MATHS EXT 1

TERM 1 TASK 2

2014

Section I :

1. From 0 to 9, there are 10 digits.
For a seven digit telephone number, the number of different numbers are $10\times9\times8\times7\times6\times5\times4 = \frac{10!}{3!}$

Hence D

2. $S_4 = \frac{4}{2}(2a+3d)$

i.e. $S_{AP} = 2(2a+3d)$

and $S_4 = \frac{a(1-r^4)}{1-r}$

i.e. $S_{GP} = \frac{a(1-r^4)}{1-r}$

But $\frac{S_{AP}}{S_{GP}} = \frac{2}{3} = \frac{4a+6d}{a(1-r^4)/(1-r)}$

But $d = r = 2$

$\therefore \frac{2}{3} = \frac{4a+12}{a(1-2^4)} \times (1-2)$

$= \frac{4a+12}{15a}$

$30a = 12a+36$

$18a = 36$

$a = 2$

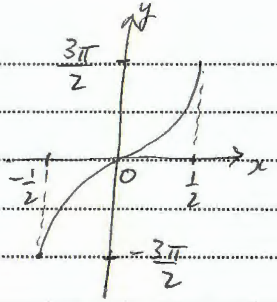
Hence $S_{AP} + S_{GP} = 2(2\times2+3\times2) + \frac{2(1-2^4)}{1-2} = 20+30 = 50$

Hence A

3. $y = 3\sin^{-1}2x$

$2x = \sin(y/3)$

$\therefore x = \frac{1}{2}\sin(y/3)$



Hence C

4. $\frac{2n!}{(n-1)!} + \frac{2(n-1)!(n-1)!}{[(n-2)!]^2}$

$= \frac{2n(n-1)!}{(n-1)!} + \frac{2(n-1)(n-2)!(n-1)(n-2)!}{(n-2)!(n-2)!}$

$= 2n + 2(n-1)(n-1)$

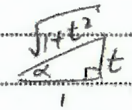
$= 2n + 2n^2 - 4n + 2$

$= 2n^2 - 2n + 2$

$= 2(n^2-n+1)$

Hence D

5. $t = \tan\alpha \Rightarrow$



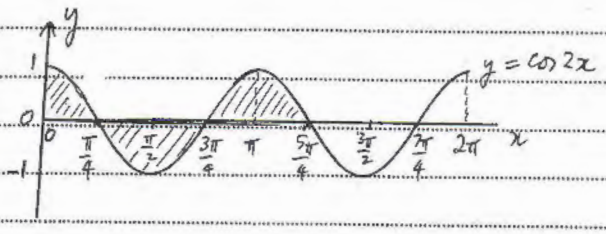
$\therefore \frac{\cos2\alpha}{1+\sin2\alpha} = \frac{\cos^2\alpha - \sin^2\alpha}{1+2\sin\alpha\cos\alpha} = \frac{\frac{1}{1+t^2} - \frac{t^2}{1+t^2}}{1+2\times\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}}}$

$= \frac{1-t^2}{1+t^2+2t}$

$= \frac{(1-t)(1+t)}{(1+t)(1+t)} = \frac{1-t}{1+t}$

Hence C

MATHEMATICS: Question 6

Suggested Solutions	Marks	Marker's Comments
SECTION II Question 6 : a) $\int \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{1+e^{2x}}\,dx$ $= \frac{1}{2}\ln(1+e^{2x}) + C$	②	① $\Rightarrow \frac{1}{2}$ ① $\Rightarrow \ln(1+e^{2x})$
b) $\int_0^{\pi/3} \cos^2 5x\,dx$ $= \int_0^{\pi/3} \frac{1}{2}(\cos 10x + 1)\,dx$ $\cos 2x = 2\cos^2 x - 1$ $\therefore \cos 10x = 2\cos^2 5x - 1$ or $\cos^2 5x = \frac{1}{2}(\cos 10x + 1)$ $= \frac{1}{2}\left[\frac{1}{10}\sin 10x + x\right]_0^{\pi/3}$ $= \frac{1}{2}\left[\frac{1}{10}\sin\frac{10\pi}{3} + \frac{\pi}{3} - \frac{1}{10}\sin 0 - 0\right]$ $= \frac{1}{2}\left[\frac{1}{10}\left(-\sin\frac{\pi}{3}\right) + \frac{\pi}{3} - 0\right]$ $= \frac{1}{2}\left[\frac{1}{10}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{3}\right]$ $= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{20}\right)$	③	① correct integration ① $\pi/6$ ① $\frac{\sqrt{3}}{40}$
c) $f(x) = 1 - 2\sin^2 x$ $= \cos 2x$ (i)  (ii) Shaded region		① shape scale x intercepts ① y intercepts and max/min values period ① shading

MATHEMATICS: Question 6		
Suggested Solutions	**Marks**	**Marker's Comments**
(iii) $A_{shaded} = 5\int_0^{\pi/4} \cos 2x \, dx$ $= 5 \times \left[\frac{1}{2}\sin 2x\right]_0^{\pi/4}$ $= 5\times\left(\frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\times 0\right)$ $= \frac{5}{2}$ $\therefore$ Area of shaded region is $2\frac{1}{2}$ unit^2	2	1 integration and "one loop" 1 correct answer
d) $S_n = 9\frac{1}{32} + 11\frac{1}{64} + 13\frac{1}{128} + \cdots$ $= (9 + 11 + 13 + \cdots) + \left(\frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots\right)$ $= \frac{n}{2}\left(2\times 9 + (n-1)2\right) + \frac{\frac{1}{32}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ $= \frac{n}{2}\left(18+2n-2\right) + \frac{1}{16}\left(1-2^{-n}\right)$ $= n^2 + 8n + \frac{1}{16}\left(1-2^{-n}\right)$	2	1 AP answer $\frac{n}{2}\left[18+2n-2\right]$ 1 GP Answer $\frac{\frac{1}{16}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$

MARKING COMMENTS

Extension I - Question 7

(a)

Response	**Mark**
14 400	2
$3! \times 5!$ (720 in a row -> multiple of 720)	1
$8! - \ldots = 36000$	1
Treated as 4 vowels and 4 consonants -> 480	1
Understand pattern only as CVCVCVCV *or* VCVCVCVC	0

(b) (i)

Response	**Mark**
Differentiated 'as a whole' to get $$\frac{1}{1+cosec^2x} \times \frac{d}{dx}\left(\frac{1}{sinx}\right)$$ *OR* $$\frac{1}{1+cosec^2x} \times \frac{d}{dx}cosecx$$	1
Correct differentiation of $\frac{1}{sinx}$ and next step to get solution	1

(b) (ii)

Response	**Mark**
Use previous result to get $-[tan^{-1}(cosec\left(\frac{\pi}{2}\right) - tan^{-1}(cosec\left(\frac{\pi}{6}\right)]$	1
Use one of two methods to get the result	1

(c) (i)

Response	**Mark**
Slope of tangent line at Q is q	1
Equation of tangent line at Q is $y = qx - aq^2$	1
Substitute in x coordinate $2ap$ ANSWER: $T(2ap, aq(2p-q))$	1

EQUATION

There are 8! possible arrangements. We need to subtract the arrangements where there are 3 or 2 consonants together.

3 consonants together

C	C	C	V	V	V	V	V

3	2	1	5	4	3	2	1

There are 6 places where the 3 consonants can be placed

$\therefore 6 \times 3! \times 5!$

2 consonants together at the start or end.

C	C	V	C/V	C/V	C/V	C/V	C/V

3	2	5	5	4	3	2	1

$\therefore 2 \times 3 \times 2 \times 5 \times 5!$

2 consonants together in the "middle". There are 5 positions for the pair of 2 consonants

V	C	C	V	C/V	C/V	C/V	C/V

$\therefore 5 \times 5 \times 3 \times 2 \times 4 \times 4!$

$\therefore 8! - (6 \times 3! \times 5! + 2 \times 3 \times 2 \times 5 \times 5! +5 \times 3 \times 2 \times 4 \times 4!)$

$= 14\,400$

(c) (ii)

Response	**Mark**
Prove that $pq=-1$ OR If this had been proved previously, there is an opportunity to gain this mark by establishing that: $q=-\frac{2a}{x}$ Needed to relate p and q to get this mark.	1
Use the relationship $pq=-1$ successfully to remove p and q and establish the required equation with x and y as variables (and NO parameters).	1

MATHEMATICS Extension 1 : Question 8

Suggested Solutions	Marks	Marker's Comments
a) (i)		
$\frac{d}{dx}(\ln x)^n = n(\ln x)^{n-1} \times \frac{1}{x}$		
$= \frac{n(\ln x)^{n-1}}{x}$	1	
(ii) $V = \pi \int_1^e y^2\,dx$		
$= \pi \int_1^e \left(\frac{\ln x}{\sqrt{x}}\right)^2 dx$		
$= \pi \int_1^e \frac{(\ln x)^2}{x}\,dx$	1	
$\frac{d}{dx}(\ln x)^n = \frac{n(\ln x)^{n-1}}{x}$		
$\therefore \int \frac{(\ln x)^{n-1}}{x} = \frac{1}{n}(\ln x)^n$		Too many students said $(\ln x)^2 = 2\ln x$ which is not true
$V = \pi \int \frac{(\ln x)^2 dx}{x} = \frac{\pi}{3}\left[(\ln x)^3\right]_1^e$	1	
$= \frac{\pi}{3}\left[(\ln e)^3 - 0\right]$		again students said $(\ln e)^3 = 3\ln e$ poor mathematics
$= \frac{\pi}{3}\ u^3$	1	

MATHEMATICS Extension 1: Question 8

Suggested Solutions	Marks	Marker's Comments
(b) $y = \ln(1+\sin x) - \ln(\cos x)$		
$y' = \dfrac{\cos x}{1+\sin x} - \left(\dfrac{-\sin x}{\cos x}\right)$	1	
$= \dfrac{\cos^2 x + \sin x + \sin^2 x}{\cos x(1+\sin x)}$		
$= \dfrac{1+\sin x}{\cos x(1+\sin x)}$		many students failed to simplify and ended up with surds
$= \dfrac{1}{\cos x}$		
at $x = \pi/3$ $\quad y' = \dfrac{1}{\cos \pi/3} = 2$	1	
$\therefore M_{normal} = -\dfrac{1}{2}$		
when $x = \dfrac{\pi}{3}$ $\quad y = \ln\left(\dfrac{1+\sin \pi/3}{\cos \pi/3}\right)$		
$= \ln(2+\sqrt{3})$	1	
$y - \ln(2+\sqrt{3}) = -\frac{1}{2}\left(x - \pi/3\right)$		
$\therefore x + 2y - \left(\pi/3 + 2\ln(2+\sqrt{3})\right) = 0$	1	

MATHEMATICS Extension 1 : Question........

Suggested Solutions	Marks	Marker's Comments
(c) (i) $a = \ln 2 \quad d = 2\ln 2$		
$S_n = \frac{n}{2}\left[2\ln 2 + (n-1)\times 2\ln 2\right]$		needed to clearly indicate $\ln 4 = 2\ln 2$ in your working
$= \frac{n}{2}(2n\ln 2)$		
$= n^2\ln 2$	1	
(ii) $T_n = a + (n-1)d$		
$= \ln 2 + (n-1)\times 2\ln 2$		
$= 2n\ln 2 - \ln 2$		
$= (2n-1)\ln 2$	1	
$S_n > 50T_n$		
$n^2\ln 2 > 50(2n-1)\ln 2$		
$n^2 - 100n + 50 > 0$		
$\therefore n < 50 - 35\sqrt{2}$ or $n > 50 + 35\sqrt{2}$	1	Had to state 100th term. least value to gain full marks.
$n > 99.5$		
least value of n is 100 since $n > 1$		
$50 - 35\sqrt{2} = 0.5$		
$50 + 35\sqrt{2} = 99.5$		

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Suggested Solutions	Marks	Marker's Comments
a) i) $y = R\cos(\theta-\alpha)$ $= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$ $\equiv \cos\theta + \sqrt{3}\sin\theta$ Equate coefficients: $\left.\begin{array}{l} R\sin\alpha = \sqrt{3} \\ R\cos\alpha = 1 \end{array}\right\}$	1	
Solve $R^2 = 4$ $R = 2 \quad (R>0)$ $\tan\alpha = \sqrt{3}$ $\alpha = \pi/3 \quad (0 \leq \alpha \leq \pi/2)$ $\therefore \quad y = 2\cos(\theta - \pi/3)$	1	
ii) $2\cos(\theta-\pi/3) = 1$ $\cos(\theta-\pi/3) = \frac{1}{2}$	1	All 3 results needed for last mark. Too many people missed it.
$\therefore \quad \theta - \pi/3 = -\pi/3, \pi/3, 5\pi/3, \ldots$ $\theta = 0, \frac{2\pi}{3}, 2\pi \quad (0 \leq \theta \leq 2\pi)$	1	
b) i) α) Let A_n be amount owing after n^{th} payment $A_1 = 20000 - M$ $A_2 = 20000 - 2M$ (no interest)	1	
β) $A_3 = (20000-2M)(1.01) - M$ $A_4 = ((20000-2M)(1.01)-M)(1.01) - M$ $= (20000-2M)(1.01)^2 - M(1.01+1)$	1	
$= (20000-2M)(1.01)^2 - 2.01M$		

X1 MATHEMATICS: Question 9 — 2013		
Suggested Solutions	**Marks**	**Marker's Comments**
ii) By inspection, $A_n = (20000 - 2M)(1.01)^{n-2} - M(1 + 1.01 + \dots + (1.01)^{n-3})$		
$= (20000-2M)(1.01)^{n-2} - M\left(\dfrac{(1.01)^{n-2}-1}{0.01}\right)$		
$\overset{*}{=} (20000-2M)(1.01)^{n-2} - 100M((1.01)^{n-2}-1)$	1	Correct generalisation
In particular, $A_{48} = 0$		
$\therefore (20000-2M)(1.01)^{46} = 100M((1.01)^{46}-1)$		
$\therefore M(102(1.01)^{46}-100) = 20000(1.01)^{46}$		
$M = \dfrac{20000(1.01)^{46}}{102(1.01)^{46}-100}$	1	manipulation
$M = \dfrac{200(1.01)^{46}}{(1.02)(1.01)^{46}-1}$	1	final answer.
iii) $M = \$516.43(24\dots)$		
$\therefore A_{24} = (20000-2M)(1.01)^{22} - 100M((1.01)^{22}-1)$		
$= \$10970.77$	1	
After extra payment $B_0 = \$8970.77$		
Amount owing after n payments (restarting count) is		
$B_n = 8970.77(1.01)^n - 100M((1.01)^n - 1)$		
Find n when $B_n = 0$		
$(1.01)^n(100M - 8970.77) = 100M$		

Suggested Solutions	Marks	Marker's Comments
$(1\cdot01)^n = \dfrac{100M}{100M - 8970\cdot77}$		
$= 1\cdot210224$		
$\therefore n\ln(1\cdot01) = \ln(1\cdot210224)$		Several people were careless at last step.
$n = \dfrac{\ln(1\cdot210224)}{\ln(1\cdot01)}$		
$= \underline{\underline{19\cdot176}}$	1	
Loan paid off on $\underline{\underline{20^{th}\text{ payment}}}$ (last payment reduced).		

3u T1 2014 Yr 12 Q10

a i) $y' = e^{-x}(2x - x^2 + 3)$

$y' = -e^{-x}(x-3)(x+1)$

SP $y' = 0$ when $x = -1$ or 3 $(e^{-x} \neq 0)$ — 1 m

Check max or min

x	-1.1	-1	0	3	3.5
y'	-1.23	0	3	0	-0.0679

$\diagdown \quad — \quad / \quad — \quad \diagdown$

rel min at $x = -1$ — 1 m

rel max at $x = 3$ — 1 m

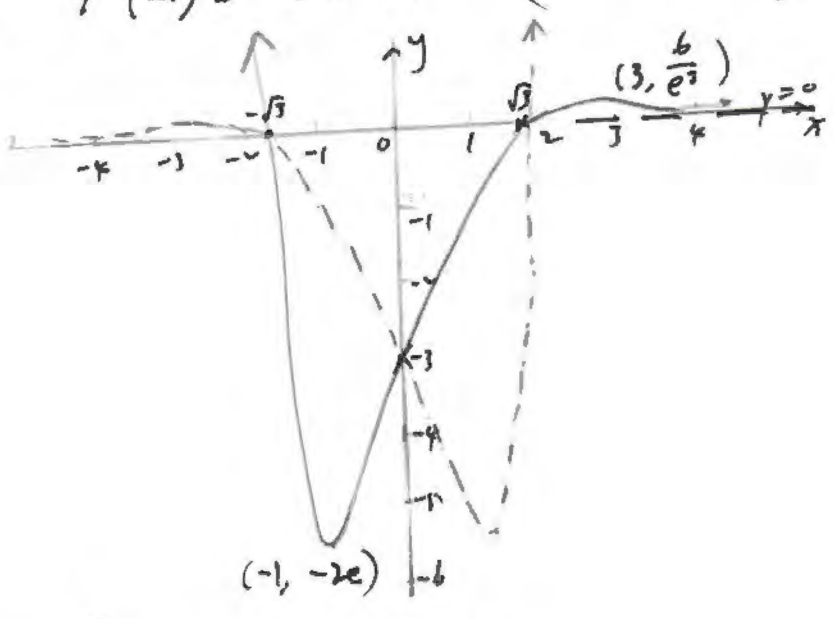
or $y'' = e^{-x}(x^2 - 4x - 1)$ — 1 m

$y''(3) = -4e^{-3} < 0$ $\therefore$ rel max at $x = 3$ (or -0.199)

$y''(-1) = 4e > 0$ rel min at $x = -1$ (or 10.87)

$\}$ 1 m

ii)



x-intercepts $x = \pm\sqrt{3}$ ✓ ✓

y-intercepts $y = -3$ ✓

horiz asymptote $y = 0$ ✓

max $(3, \frac{6}{e^3})$ ✓

min $(-1, -2e)$ ✓

shape ✓

scale ✓

Every 2 ticks get 1 m

no $\frac{1}{2}$ mark

iii) $e^{x}(x^2 - 3) \geq e^{-x}(x^2 - 3)$

$-\sqrt{3} \leq x \leq 0$ or $x \geq \sqrt{3}$

sketch $e^{x}(x^2 - 3)$ — 1 m

$-\sqrt{3} \leq x \leq 0$ or $x \geq \sqrt{3}$ — 1 m

(some use 'and' instead of 'or' don't get this mark)

b) $f'(x) = \lim_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h}$

$= \lim_{h\to 0} \dfrac{\sin x \cosh + \cos x \sin h - \sin x}{h}$

$= \lim_{h\to 0} \dfrac{\sin x(\cos h - 1)}{h} - \lim_{h\to 0}\left(\dfrac{\sin h}{h}\right)\cos x$ 1m

$= \lim_{h\to 0} \sin x\left[\dfrac{(\cos h - 1)(\cos h + 1)}{h\ (\cos h + 1)}\right] - 1 \times \cos x$ v. poorly done most students get 1m.

$= \lim \sin x \dfrac{(\cos^2 h - 1)}{h(\cos h + 1)} - \cos x$

$= \lim_{h\to 0} (\sin x)\left(\dfrac{-\sin h}{(\cos h + 1)}\right)\dfrac{(\sin h)}{h} - \cos x$

$= \sin x \cdot \dfrac{0}{2} \cdot 1 - \cos x$ 1m

$\dfrac{d}{dx}(\sin x) = \cos x$ #

or

$f'(x) = \lim_{h\to 0} \dfrac{\sin(x + \frac{h}{2} + \frac{h}{2}) - \sin(x + \frac{h}{2} - \frac{h}{2})}{h}$ 1m

$= \lim_{h\to 0} 2\cos(x + \frac{h}{2}) \dfrac{\sin\frac{h}{2}}{h}$

$= \lim_{h\to 0} \cos(x + \frac{h}{2}) \cdot \left(\dfrac{\sin\frac{h}{2}}{h/2}\right)$

$= \cos x \cdot 1$ 1m

$= \underline{\underline{\cos x}}$

some use inverse trig don't get any marks.

MATHEMATICS EXT 1: Question 11.a (ii)		
Suggested Solutions	**Marks**	**Marker's Comments**
(ii) $S = r\theta$ and $\theta = 2\alpha = 2\left[\tan^{-1}\frac{h}{r}\right]$ $\boxed{\frac{ds}{dt} = \frac{ds}{dh}\cdot\frac{dh}{dt}}$ First get $\frac{ds}{dh} = \frac{ds}{d\theta}\cdot\frac{d\theta}{dh}$ $s = r\theta$ $\frac{ds}{d\theta} = r$ $\frac{d\theta}{dh} = 2\tan^{-1}\frac{h}{r}$ $= 2\times\frac{1}{2}\left(\frac{1}{1+\left(\frac{h}{r}\right)^2}\right)$ $= \frac{2r}{r^2+h^2}$ $\therefore \frac{ds}{dh} = \frac{2r^2}{h^2+r^2}$ $\therefore \frac{ds}{dt} = \frac{2r^2}{h^2+r^2}\times 2$ $= \frac{4r^2}{h^2+r^2}$ Now substitute $h = \frac{r}{2}$. $\therefore \frac{ds}{dt} = \frac{4r^2}{r^2+\left(\frac{r}{2}\right)^2}$ $= \frac{4r^2}{r^2+\frac{r^2}{4}}$ $= \frac{4r^2\times 4}{4r^2+r^2}$ $= \frac{16r^2}{5r^2}$ $= \frac{16}{5}$ $\therefore$ rate of shadow moving is $\frac{16}{5}$ m/s	✓ $\frac{ds}{dh}$ ✓ $\frac{dh}{dt}$ ✓ $\frac{ds}{dt}$ ✓ 1 mark subst	Several methods $s = r\theta$ $\therefore \frac{ds}{d\theta} = r$ Note 1 r is a constant so you cannot do $\frac{ds}{dr}$ or $\frac{dh}{dr}$ or $\frac{dr}{dt}$. has no meaning Note 2 you cannot substitute $h = \frac{r}{2}$ before you differentiate 1 mark was given for $\frac{dh}{dt} = 2$. One also used $\frac{dh}{dt} = -2$. $\therefore$ got $\frac{ds}{dt}$ as $\frac{-4r^2}{h^2+r^2}$. $= \frac{-16}{5}$ m/s. Those who got $2\tan^{-1}(2)$ or $4\tan^{-1}\left(\frac{1}{2}\right)$ or $8\tan^{-1}\left(\frac{1}{2}\right)$ used $\frac{ds}{dr}$ or substitute before differentiating

MATHEMATICS EXT 1: Question 11 a (ii) continued.

Suggested Solutions	Marks	Marker's Comments

Method 2

$s = r\theta$

$\dfrac{ds}{d\theta} = r$

$$\frac{ds}{dt} = \frac{ds}{d\theta} \times \frac{d\theta}{dt}$$

$$= \frac{ds}{d\theta} \times \frac{d\theta}{dh} \times \frac{dh}{dt}.$$

$$= r \times \frac{2r}{r^2+h^2} \times 2$$

$$= \frac{4r^2}{r^2+h^2}$$

then sub $h = \frac{r}{2}$

Some substituted. $\dfrac{d\theta}{dh} = \dfrac{8}{5r}$ after using $h = r\tan\frac{\theta}{2} \therefore \frac{dh}{d\theta} = \frac{1}{2}r\sec^2\theta$ and got $\frac{5}{4}$ as $\sec^2\theta$

Marker's Comments:

$$\frac{d\theta}{dh} = \frac{2\cos^2\frac{\theta}{2}}{r}$$

$$\frac{d\theta}{dh} = 2\tan^{-1}\frac{h}{r}$$

$$= 2 \times \frac{1}{r} \times \frac{1}{1+\left(\frac{h}{r}\right)^2}$$

$$= \frac{2r}{r^2+h^2}$$

Method 3

$$\frac{ds}{dt} = \frac{ds}{d\alpha} \times \frac{d\alpha}{dh} \times \frac{dh}{dt}.$$

$$= 2r \times \frac{r}{h^2+r^2} \times 2$$

Marker's Comments:

$$\frac{d\alpha}{dh} = \frac{d}{dh}\left(\tan^{-1}\frac{h}{r}\right)$$

$$= \frac{1}{1+\frac{h^2}{r^2}} \times \frac{1}{r}$$

$$= \frac{r}{r^2+h^2}$$

$$\frac{ds}{dt} = \frac{ds}{dh} \times \frac{dh}{dt}$$

$$\frac{ds}{dt} = \frac{ds}{d\theta} \times \frac{d\theta}{dh} \times \frac{dh}{dt}.$$

Similar to method 2.

$$\therefore \frac{ds}{dh} = r \times 2 \times \frac{\cos^2\frac{\theta}{2}}{r} \times 2.$$

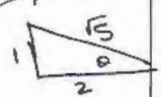
$$= 4\cos^2\frac{\theta}{2}.$$

$$= 4\cos^2\left[\frac{2\tan^{-1}\left(\frac{1}{2}\right)}{2}\right]$$

$$= 4\cos^2\left(\tan^{-1}\frac{1}{2}\right)$$

$$= 4 \times \frac{4}{5}$$

$$= 16/5$$



$\tan\theta = \frac{1}{2}$

$\therefore \cos\theta = \frac{2}{\sqrt{5}}$

and $\cos^2\theta = \frac{4}{5}$

Marker's Comments:

$$\tan^{-1}\left(\frac{h}{r}\right) = \frac{\theta}{2}$$

$$\tan\frac{\theta}{2} = \frac{h}{r}$$

$$h = r\tan\frac{\theta}{2}$$

$$\frac{dh}{d\theta} = \frac{r}{2}\sec^2\frac{\theta}{2}$$

$$\therefore \frac{d\theta}{dh} = \frac{2\cos^2\frac{\theta}{2}}{r}.$$

MATHEMATICS EXT 1: Question 11

Suggested Solutions	Marks	Marker's Comments
Some values used were:		
① $\frac{dh}{dt} = 2$		
② $\frac{ds}{d\theta} = r$		
③ $\frac{d\theta}{dh} = \frac{8}{5r} = \frac{2r}{r^2+h^2} = \frac{2\cos^2\frac{\theta}{2}}{r}$		
④ $\frac{ds}{d\alpha} = 2r$		
⑤ $\frac{d\alpha}{dh} = \frac{r}{r^2+h^2}$ or $\frac{d}{dh}\left(\tan^{-1}\left(\frac{h}{r}\right)\right)$		
⑥ $\frac{d\theta}{dt} = \frac{4r}{r^2+h^2}$		
⑦ $\frac{d\alpha}{dt} = \frac{2r}{r^2+h^2}$		
⑧ $\frac{ds}{dh} = \frac{d\theta}{dh} \times \frac{ds}{d\theta}$ or $\frac{ds}{d\alpha} \times \frac{d\alpha}{dh}$.		

MATHEMATICS EXT 1: Question 11

Suggested Solutions	Marks	Marker's Comments
(b) - Mathematical Induction		For step 1
To prove $4^n \geq 1+3n$, $n = 1, 2, 3 \ldots$	*	It was necessary for students to prove LHS $\geq$ RHS and write the statement down.
step 1 test $n=1$ LHS $= 4^1 = 4$ RHS $= 1+3(1) = 4$ $\therefore 4^n = 1+3n$ for $n=1$ you could test for $n=2$ LHS $= 4^2 = 16$ RHS $= 1+3(2) = 7$ $\therefore$ LHS $>$ RHS		
step 2 Assume true for $n=k$ where $k \in Z^n$ i.e $4^k \geq 1+3k$.	*	step 2, or 3 one had to mention $k \in Z^n$ or positive integers.
Step 3 prove true for $n=k+1$ ie prove $4^{k+1} \geq 1+3(k+1)$ $\geq 4+3k$. Now $4^{k+1} = 4^k \times 4^1$ $4 \times 4^k \geq 4(1+3k)$ (By assumption) $4^{k+1} \geq 4+12k$. $4^{k+1} \geq 4+3k+9k$. since $9k > 0$ for $k \in Z^+$, it stands that $4^{k+1} \geq 4+3k$.	* *	for step 3 One had to write by assumption if you used step 2 i.e $4^k \geq (1+3k)$ For step 3 simplify RHS and need to go LHS and not use what you trying to prove.
Step 4 $\therefore$ reason that $4^{k+1} \geq 4+3k$ Hence by Proof of mathematical Induction statement is true for all, $n \geq 1$, $n \in Z^+$		
Alternatively step 2 - $4^k \geq 1+3k$ can be written as $4^k - 3k - 1 \geq 0$ similarly $4^{k+1} - 3(k+1) - 1 \geq 0$ ie $4^{k+1} - 3k - 4 \geq 0$ $= 4(4^k - 3k - 1) + 9k \geq 0$.		

MATHEMATICS Extension 1 : Question 12

Suggested Solutions	Marks	Marker's Comments
(a) $x = \sec\theta \qquad \therefore \theta = \sec^{-1} x$		
$\frac{dx}{d\theta} = \sec\theta . \tan\theta$		
when $x = 2$, $\theta = \sec^{-1}(2)$, $\theta = \pi/3$		
when $x = \sqrt{2}$, $\theta = \sec^{-1}(\sqrt{2})$, $\theta = \pi/4$		
$I = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta \sqrt{\sec^2\theta - 1}} . \sec\theta \tan\theta \, d\theta$	1mk.	
$= \int_{\pi/4}^{\pi/3} \frac{\sec\theta \tan\theta \, d\theta}{\sec^3\theta . \tan\theta}$		
$= \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta} \, d\theta$		* very sad to see the number of students who integrated $\frac{1}{\sec^2\theta}$ as $\tan\theta$ = 2mks
$= \int_{\pi/4}^{\pi/3} \cos^2\theta \, d\theta$	1 mk.	
$= \frac{1}{2}\int_{\pi/4}^{\pi/3} (\cos 2\theta + 1) \, d\theta$		
$= \frac{1}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\pi/4}^{\pi/3}$	1	
$= \frac{1}{2}\left[\frac{1}{2}\sin \frac{2\pi}{3} + \frac{\pi}{3} - \frac{1}{2}\sin\frac{\pi}{2} - \frac{\pi}{4}\right]$		* lost the mk if you didn't know the value of $\sin\pi/3$ and $\sin\pi/2$
$= \frac{1}{2}\left(\frac{\sqrt{3}}{4} + \frac{\pi}{3} - \frac{1}{2} - \frac{\pi}{4}\right)$	1	
$= \frac{\pi}{24} + \frac{3\sqrt{3}}{24} - \frac{1}{4}$		
(b) (i) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$		1 mk (right or wrong).

MATHEMATICS Extension 1 : Question 12

Suggested Solutions	Marks	Marker's Comments
(b)(ii) let $A=(n+1)\theta \quad B=n\theta \quad n\in J^+$		← most students didn't put this in :(
$\therefore \tan\left[(n+1)\theta - n\theta\right] = \dfrac{\tan(n+1)\theta - \tan n\theta}{1+\tan(n+1)\theta \cdot \tan n\theta}$		1
$\tan(n\theta+\theta-n\theta) = \dfrac{\tan(n+1)\theta - \tan n\theta}{1+\tan(n+1)\theta\tan n\theta}$		
$\therefore \tan\theta = \dfrac{\tan(n+1)\theta - \tan n\theta}{1+\tan(n+1)\theta\tan n\theta}$		1 *Some students tried to prove the given identity but fudged the trig = 0mks.
$\therefore 1+\tan(n+1)\theta\tan n\theta = \dfrac{\tan(n+1)\theta - \tan n\theta}{\tan\theta}$		
*Some students set it up like a proof, using LHS = ... but then they used the RHS to prove that it was the RHS!!! Max 1mk.		
(iii)		
LHS = $\tan\theta\tan2\theta + \tan2\theta\tan3\theta + \ldots\ldots + \tan(n\theta).\tan(n+1)\theta$		
(1) $= \dfrac{\tan2\theta-\tan\theta}{\tan\theta} - 1 + \dfrac{\tan3\theta-\tan2\theta}{\tan\theta} - 1 + \ldots\ldots + \dfrac{\tan(n+1)\theta-\tan n\theta}{\tan\theta}$ from (ii)		
(1) $= \dfrac{1}{\tan\theta}\left(\tan2\theta-\tan\theta+\tan3\theta-\tan2\theta+\ldots\ldots+\tan(n+1)\theta-\tan n\theta\right) - n$		
(1) $= \dfrac{1}{\tan\theta}\left(\tan(n+1)\theta - \tan\theta\right) - n$		
(1) $= \cot\theta\tan(n+1)\theta - 1 - n$		*A lot of fudging badly set out,
$= \cot\theta\tan(n+1)\theta - (n+1)$		*Some did it by Induction but didn't set it out properly or fudged it
= RHS.		
*Some students never wrote out the "n^{th} term" but then it suddenly appeared on their second last line. It looks like fudging ⇒ max 2 marks!!		