QUESTION 1 (Use a SEPARATE writing booklet)

a) Find $\lim_{x \to 0} \frac{sin(5x)}{2x}$ **1**

b) A is the point (-2, 5) and B is the point (2, -3). If P is a point on AB which divides AB in the ratio 3:1, find the coordinates of P. **2**

c) Write down the exact value of

$$\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right)$$ **2**

d) Find:

i) $$\int \left(x + \frac{1}{x}\right)^2 dx$$ **2**

ii) $$\int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)dx$$ **3**

e) Find the gradient of the tangent to

$$y = \tan^{-1}\left(\frac{x}{3}\right)$$

at the point where

$$x = \sqrt{3}$$ **2**

QUESTION 2 (Use a SEPARATE writing booklet)

a) i) Express $5sinx + 3cosx$ in the form $Rsin(x + \alpha)$ where $R \geq 0$ and α is acute and expressed in degrees, rounded off to the nearest degree. **2**

ii) Use this to solve the equation $5sinx = 4 - 3cosx$ for $0^{\circ} \leq x \leq 360^{\circ}$ **2**

b) i) Write down the domain and range of $y = -2\sin^{-1}\left(\frac{x}{2}\right)$ **2**

ii) Sketch the graph of $y = -2\sin^{-1}\left(\frac{x}{2}\right)$ **1**

iii) Show that the equation $2\sin^{-1}\left(\frac{x}{2}\right) + 2x + 1 = 0$ has a zero between $x = -\frac{1}{2}$ and $x = 0$. **1**

iv) Use one application of Newton's Method, using $x = -0.4$ as your initial value, to find a better estimate for this zero (give your answer accurate to three decimal places). **2**

c) i) Find $\int \frac{5x}{25 + x^2}\,dx$ **1**

ii) Find $\int \frac{5}{25 + x^2}\,dx$ **1**

QUESTION 3 (Use a SEPARATE writing booklet)

a) Solve for x: $\log_2 x - 3\log_x 2 = 2$ **2**

b) Solve for x: $\frac{1}{x} - 2 \geq \frac{4x}{x+1}$ **4**

c) The graph of $y = \sin x$ is rotated around the X axis between the values $x = 0$ and $x = \pi$. Find the volume of the solid which is thus generated. **3**

d) Use the substitution $u = 2 + x$ to evaluate

$$\int \frac{x}{\sqrt{2+x}} dx$$

3

Question 4 (Use a SEPARATE writing booklet)

a) If $log_{10} 2 = a$ and $log_{10} 3 = b$ find each of the following in terms of *a* and *b:*

i) $log_{10} 6$ **1**

ii) $log_{10} \frac{5}{3}$ **2**

b) Consider the function

$$f(x) = \frac{x}{x+2}$$

i) Show that $f'(x) > 0$ for all x in the domain and hence explain why the function has an inverse function. **2**

ii) State the equation of the horizontal asymptote of $y = f(x)$ **1**

iii) Without any further calculus, sketch the graph of $y = f(x)$ **1**

iv) Find an equation for $y = f^{-1}(x)$ **2**

v) Give the domain of $f^{-1}(x)$ **1**

vi) Find the point(s) where $f(x)$ and $f^{-1}(x)$ intersect. **2**

Question 5 (Use a SEPARATE writing booklet)

a) Find the quotient and remainder when $P(x) = x^4 - 2x^3 + 9x - 4$ is divided by $x^2 - 3x + 1$ **2**

b) If $g(x) = 2x^3 - ax^2 + (a + b)x + 10$ is divided by $x + 2$ the remainder is $b + 3$. Determine b in terms of a. **2**

c) When the polynomial $P(x)$ is divided by $(x - 2)(x + 1)$ it produces a remainder of $-2x + 8$. Find the remainder when $P(x)$ is divided by $x - 2$ only. **2**

d) The polynomial $P(x) = x^3 - 2x^2 + kx + 24$ has roots α, β and γ.

Two of the roots are equal but opposite in sign. Find the third root and hence the value of k. **3**

e) Prove by Mathematical Induction that $4^n + 14$ is divisible by 6 for all $n \geq 1$ **3**

Question 6 (Use a SEPARATE writing booklet)

a) i) Draw a neat sketch of $y = 2\log(x-2)$ **2**

ii) Find the area contained by the curve, the X-axis and the line $x = 5$ **3**

b) The tangents at $P(2ap\ ,\ ap^2)$ and $Q(2aq\ ,\ aq^2)$ on the parabola $x^2 = 4ay$ meet at T. M is the midpoint of PQ.

i) Show that the equation of the tangent at P is $y = px - ap^2$ **2**

ii) Find the coordinates of T **2**

iii) Prove that MT is parallel to the Y-axis **1**

iv) Show that the coordinates of the midpoint R of MT are

$$R\left(a(p+q)\ ,\ \frac{apq + \frac{a}{2}(p^2+q^2)}{2}\right)$$ **2**

v) Find the equation of the locus of R. **2**

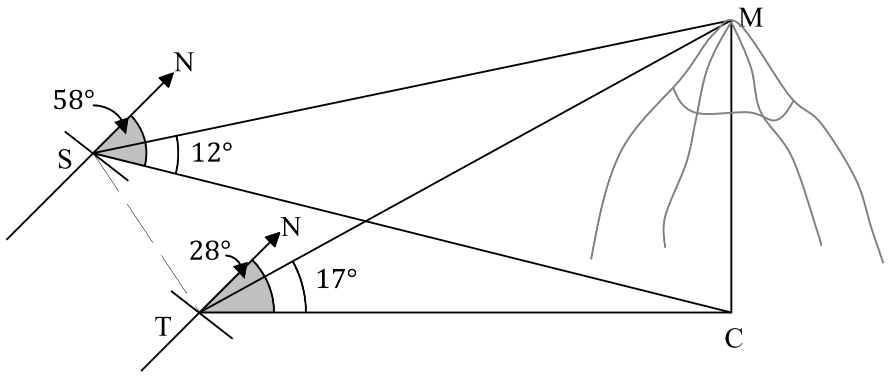
Question 7 (Use a SEPARATE writing booklet)

a) i) Prove that $2\sin 5\theta \cos 4\theta - \sin 9\theta = \sin \theta$ **2**

ii) Hence find $\int \sin 5\theta \cos 4\theta \, d\theta$ **2**

b) The intrepid TV explorer Tiger Bakes is lost in the Amazon. Luckily, he has radio contact with his support team and he knows how to use ancient Inca techniques to measure compass bearings and angles. He can see the summit of Cerro Toro, a nearby mountain which he knows to be 1200 metres high. It is on a bearing of 28° from his position and the angle of inclination is 17°. He radios this information to his support team. The support team are delighted because they can also see the summit of Cerro Toro on a bearing of 58°, and from their position it has an angle of elevation of 12°.

The diagram below represents a 3D view of the situation. In the diagram, T represents Tiger's position, S is the position of his support team, C is the foot of Cerro Torro and M is the peak.



i) Draw an aerial view (bird's eye view) of the situation and hence show that $\angle SCT = 30°$ **2**

ii) Calculate the lengths of SC and TC accurate to 2 decimal places. **2**

iii) Find the distance from S to T. **2**

iv) Find the bearing of Tiger from his support team. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{ if } n < 0.$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0.$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0.$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left\{ x + \sqrt{(x^2 - a^2)} \right\}, \; |x| > |a|$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left\{ x + \sqrt{(x^2 + a^2)} \right\}$$

NOTE: $\ln x = \log_e x, \; x > 0.$

YEAR 12 PRETRIALS EXT. 1 2011 SOLUTIONS

QUESTION 1.

a) $\lim_{x \to 0} \frac{\sin 5x}{2x} = \frac{5}{2} \lim_{x \to 0} \frac{\sin 5x}{5x} = \frac{5}{2}$

b) $P\left(\frac{6-2}{4}, \frac{-9+5}{4}\right) = P(1, -1)$

c) $\sin^{-1}\left(\sin \frac{7\pi}{6}\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

d) i) $\int \left(x + \frac{1}{x}\right)^2 dx = \int \left(x^2 + 2 + x^{-2}\right) dx = \frac{x^3}{3} + 2x - \frac{1}{x} + C$

ii) $\int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx = \int_0^{\pi/4} \cos 2x \, dx = \frac{1}{2}\left[\sin 2x\right]_0^{\pi/4}$

$= \frac{1}{2}(1) - \frac{1}{2}(0) = \frac{1}{2}$

e) $\frac{dy}{dx} = \frac{1}{3}\left(\frac{1}{1+\left(\frac{x}{3}\right)^2}\right) = \frac{1}{3 + x^2/3}$

When $x = \sqrt{3}$, $\frac{dy}{dx} = \frac{1}{3 + 3/3} = \frac{1}{4}$

QUESTION 2

a) i) $R = \sqrt{34}$ so $\sqrt{34}(\sin x \cos\alpha + \cos x \sin\alpha) = 5\sin x + 3\cos x$

$\therefore \sqrt{34}\cos\alpha = 5$ and $\sqrt{34}\sin\alpha = 3$

$\therefore \cos\alpha = \frac{5}{\sqrt{34}}$

$\therefore \alpha \approx 31^\circ$

$\therefore 5\sin x + 3\cos x \approx \sqrt{34}\sin(x + 31^\circ)$

ii) Hence $\sqrt{34}\sin(x+31^\circ) = 4$

$\therefore \sin(x+31^\circ) = \frac{4}{\sqrt{34}}$

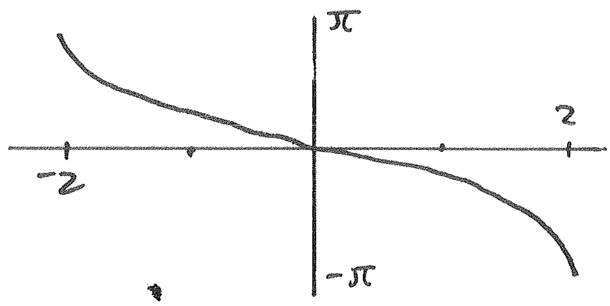
$\therefore x + 31^\circ = 43^\circ$ or $x + 31^\circ = 137^\circ$

$\therefore x = 12^\circ$ or $x = 106^\circ$

b) i) Domain : $-1 \le \frac{x}{2} \le 1$ so $-2 \le x \le 2$

Range : $-\pi \le y \le \pi$

ii)



iii) $f(-\frac{1}{2}) = 2\sin^{-1}(-\frac{1}{2}) + 2(-\frac{1}{2}) + 1 \approx -0.5 < 0$

$f(0) = 2\sin^{-1}(0) + 2(0) + 1 = 1 > 0$

$\therefore$ There is a zero between $-\frac{1}{2}$ and 0

iv) $f'(x) = \frac{1}{\sqrt{1 - \frac{x^2}{4}}} + 2$ $\qquad x_1 = -0.4 - \frac{f(-0.4)}{f'(-0.4)}$

$= -0.4 - \frac{-0.203}{3.021}$

$= -0.333$

c) i) $\int \frac{5x}{25+x^2}\,dx = \frac{5}{2}\int \frac{2x}{25+x^2}\,dx = \frac{5}{2}\ln(25+x^2) + c$

ii) $\int \frac{5}{25+x^2}\,dx = 5\int \frac{1}{25+x^2}\,dx = \frac{1}{5}\cdot 5 \tan^{-1}\frac{x}{5} + c$

$= \tan^{-1}\frac{x}{5} + c$

QUESTION 3

a) $\log_2 x - \dfrac{3}{\log_2 x} - 2 = 0$

$\therefore (\log_2 x)^2 - 2\log_2 x - 3 = 0$

$\therefore (\log_2 x - 3)(\log_2 x + 1) = 0$ (OR BY SUBSTITUTION)

$\therefore \log_2 x = 3$ or $\log_2 x = -1$

$\therefore x = 8$ or $x = \frac{1}{2}$

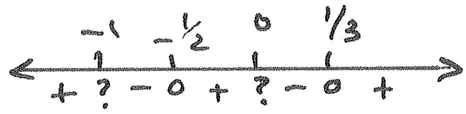
b) $\dfrac{1}{x} - 2 - \dfrac{4x}{x+1} \geq 0$

$\therefore \dfrac{x+1-2x(x+1)-4x^2}{x(x+1)} \geq 0$

$\therefore \dfrac{-6x^2 - x + 1}{x(x+1)} \geq 0$

$\therefore \dfrac{6x^2 + x - 1}{x(x+1)} \leq 0$

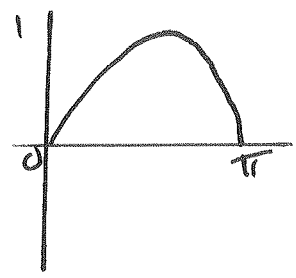
$\therefore \dfrac{(3x-1)(2x+1)}{x(x+1)} \leq 0$



$\therefore -1 < x \leq -\frac{1}{2}$ or $0 < x \leq \frac{1}{3}$

c) $V = \pi \int_0^{\pi} \sin^2 x \, dx$

$= \pi \cdot \frac{1}{2} \int_0^{\pi} (1 - \cos 2x)\, dx$



$= \frac{\pi}{2} \left[x - \frac{1}{2} \sin x \right]_0^{\pi}$

$= \frac{\pi}{2} \left[(\pi - 0) - (0 - 0) \right]$

$= \frac{\pi^2}{2}$ units^3

d) $u = 2 + x$

$\therefore du = dx$ and $x = u - 2$

$$\therefore \int \frac{x}{\sqrt{2+x}}\,dx = \int \frac{u-2}{\sqrt{u}}\,du = \int (u-2)\,u^{-1/2}\,du$$

$$= \int \left(u^{1/2} - 2u^{-1/2}\right) du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}} + C$$

QUESTION 4

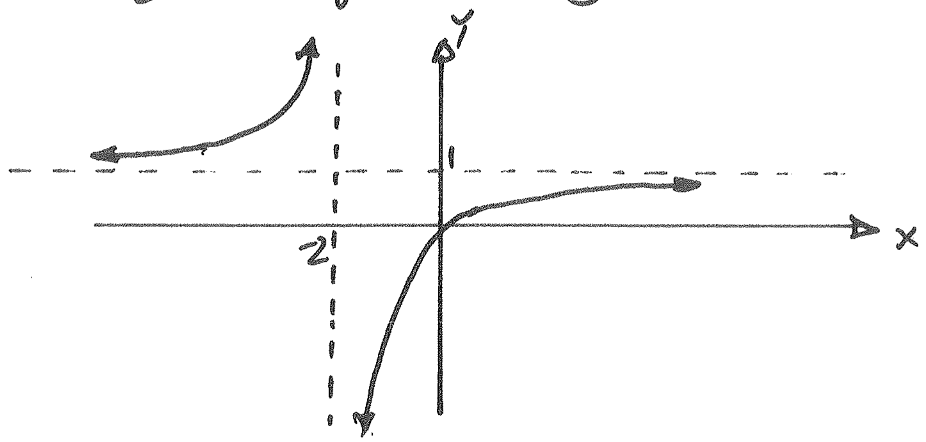
a) i) $\log_{10} 6 = \log_{10} 2 + \log_{10} 3 = a + b$

ii) $\log_{10} \frac{5}{3} = \log_{10} 10 + \log_{10} 2 - \log_{10} 3 = 1 + a - b$

b) i) $f'(x) = \frac{(x+2) - x}{(x+2)^2} = \frac{2}{(x+2)^2} > 0$ for all x. This means that f is strictly increasing and it therefore has an inverse function.

ii) $f(x) = \frac{1}{1 + \frac{2}{x}}$ for $x \neq 0$. As $x \to \pm\infty$, $f(x) \to 1$.
$\therefore$ the horizontal asymptote is $y = 1$

iii)



iv) Put $x = \frac{y}{y+2}$

$\therefore xy + 2x = y$

$\therefore xy - y = -2x$

$\therefore y(x-1) = -2x$

$\therefore y = \frac{-2x}{x-1}$ or $\frac{2x}{1-x}$

v) $x \neq 1$, $x \in \mathbb{R}$.

vi) They intersect on the line $y = x$

$\therefore x = \frac{x}{x+2}$

$\therefore x^2 + 2x = x$

$\therefore x^2 + x = 0$

$\therefore x(x+1) = 0$ so they intersect at $(0,0)$ and $(-1,-1)$

QUESTION 5

a)
$$\begin{array}{r} x^2 + x + 2 \\ x^2 - 3x + 1 \overline{\big) x^4 - 2x^3 + 0x^2 + 9x - 4} \\ \underline{x^4 - 3x^3 + x^2} \\ x^3 - x^2 + 9x \\ \underline{x^3 - 3x^2 + x} \\ 2x^2 + 8x - 4 \\ \underline{2x^2 - 6x + 2} \\ 14x - 6 \end{array}$$

(x^2+x+2) rem $(14x-6)$

b) $g(-2) = -16 - 4a - 2(a+b) + 10 = b+3$

$\therefore \quad -3b = 6a+9$

$\therefore \quad b = -2a-3$

c) $P(x) = (x-2)(x+1)Q(x) - 2x + 8$

$\therefore P(2) = 0 - 4 + 8 = \underline{4}$

d) $\alpha+\beta+\gamma = 2$

$\alpha\beta\gamma = -24$

Say roots are $\alpha, -\alpha, \gamma$. $\therefore \alpha - \alpha + \gamma = 2 \quad \therefore \gamma = 2 \quad \therefore -\alpha^2 = -12$

$\therefore k = -\alpha^2 + \alpha\gamma - \alpha\gamma = -\alpha^2 = -12$ (or by multiplying out)

e) If $n=1$, $4^n + 14 = 18 = 3 \times 6$

$\therefore$ Statement is true for $n=1$.

Assume true for some $n=k$.

$\therefore 4^k + 14 = 6p$ for some integer p.

$\therefore 4^{k+1} + 14 = 4 \cdot 4^k + 14$ (OR ALTERNATIVE)

$= 3 \cdot 4^k + 4^k + 14$

$= 3 \cdot 4^k + 6p$

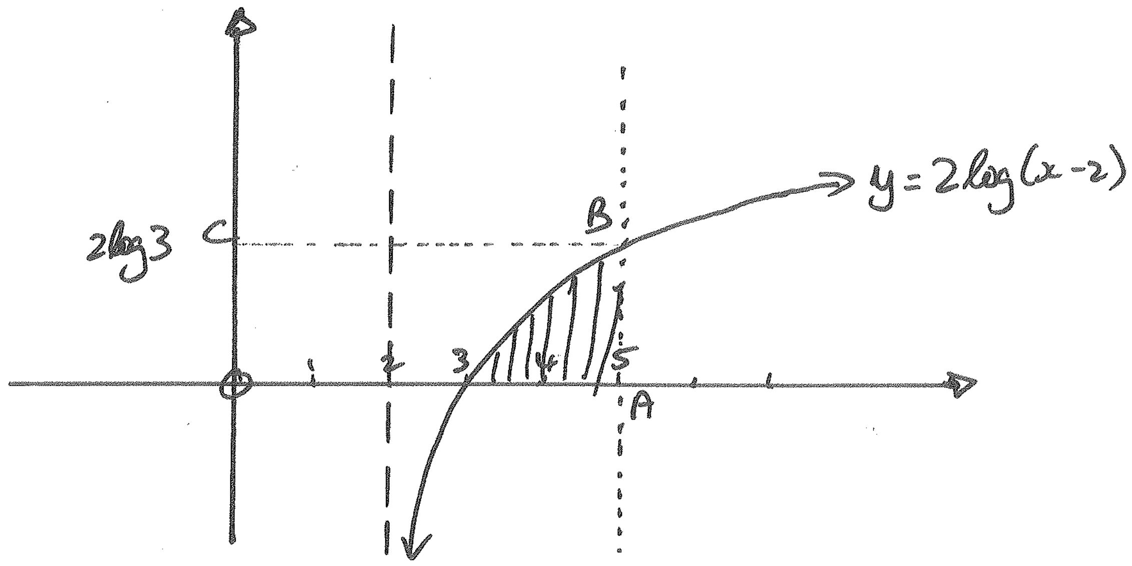
$= 6 \cdot 2^{2k-1} + 6p = 6(2^{2k-1} + p)$ which is a multiple of 6.

Hence truth for $n=k$ implies truth for $n=k+1$

But formula is true for $n=1$ $\therefore$ true for all $n \geq 1$.

QUESTION 6

a) i)



ii) Area rectangle OABC $= 5 \times 2\log 3 = 10\log 3$

If $y = 2\log(x-2)$ then $e^y = (x-2)^2$

$\therefore e^{y/2} + 2 = x$

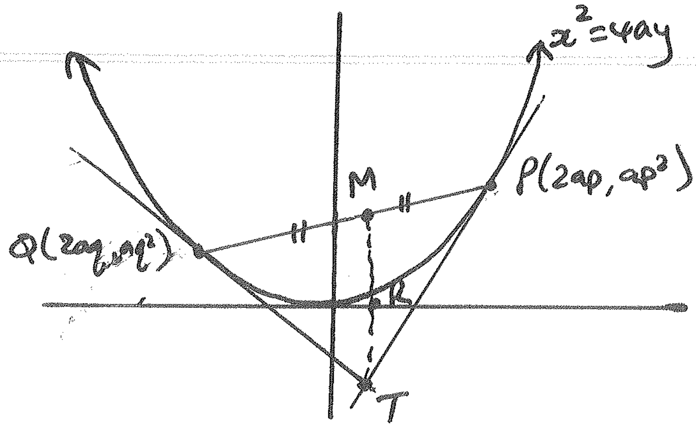
$\therefore$ Required area is $10\log 3 - \int_0^{2\log 3} (e^{y/2} + 2)\,dy$

$= 10\log 3 - \left[2e^{y/2} + 2y\right]_0^{2\log 3}$

$= 10\log 3 - (6 + 4\log 3 - 2)$

$= 6\log 3 - 4$

b) i) Gradient of tangent at P is p



$\therefore$ Equation is

$y - ap^2 = p(x - 2ap)$

$\therefore y = px - 2ap^2 + ap^2$

$\therefore y = px - ap^2$

ii) Equation of Tangent at Q is

$y = qx - aq^2$

At intersection, $px - ap^2 = qx - aq^2$

$\therefore x(p-q) = ap^2 - aq^2$

$\therefore x = \frac{a(p+q)(p-q)}{p-q} = a(p+q)$

$\therefore y = p(a)(p+q) - ap^2 = apq + ap^2 - ap^2 = apq$

$\therefore T(a(p+q), apq)$

iii) $M\left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

M and T have the same X-coordinate

Hence MT is parallel to Y-axis

iv) X-coordinate of R is $a(p+q)$ (MT is vertical)

Y-coordinate of R is $\frac{1}{2}\left(\frac{a(p^2+q^2)}{2} + apq\right) = \frac{apq + \frac{a}{2}(p^2+q^2)}{2}$.

v) Y-coordinate of R can be simplified as $\frac{\frac{a}{2}(p^2+q^2+2pq)}{2}$

$= \frac{a}{4}(p+q)^2$ and $x = a(p+q)$ so $p+q = \frac{x}{a}$

$\therefore$ by eliminating p and q, $y = \frac{a}{4} \cdot \frac{x^2}{a^2} = \frac{x^2}{4a}$

$\therefore$ locus is $x^2 = 4ay$ which is the original parabola!

QUESTION 7

a) i) LHS is $2\sin 5\theta\cos 4\theta - \sin(5\theta + 4\theta)$

$= 2\sin 5\theta\cos 4\theta - \sin 5\theta\cos 4\theta - \cos 5\theta\sin 4\theta$

$= \sin 5\theta\cos 4\theta - \cos 5\theta\sin 4\theta$

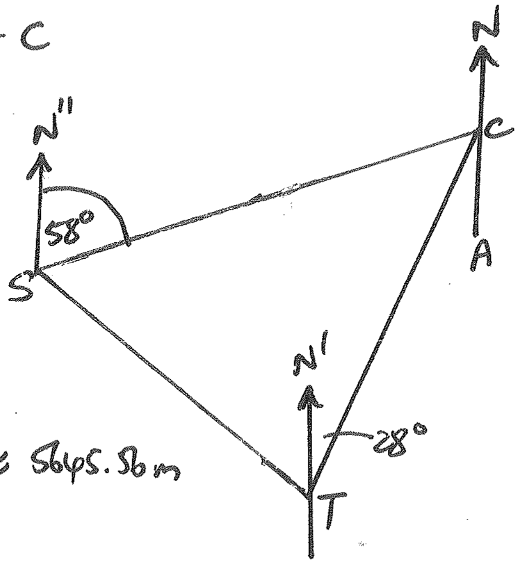
$= \sin(5\theta - 4\theta)$

$= \sin\theta \quad = \text{RHS}.$

ii) $\int \sin 5\theta\cos 4\theta\, d\theta = \frac{1}{2}\int(\sin 9\theta + \sin\theta)\,d\theta$

$= \frac{1}{2}\left(\frac{-1}{9}\right)\cos 9\theta - \frac{1}{2}\cos\theta + c$

$= -\left(\cos 9\theta + \frac{1}{2}\cos\theta\right) + c$



b) i) From the diagram,

$\angle SCA = 58°$ (alternate to $\angle N''SC$)

$\angle TCA = 38°$ (alternate to $\angle N'TC$)

$\therefore \angle SCT = 30°$ by subtraction.

ii) $\frac{SC}{MC} = \cot 12° \quad \therefore SC = 1200\cot 12° \approx 5645.56\text{ m}$

$\frac{TC}{MC} = \cot 17° \quad \therefore TC = 1200\cot 17° \approx 3925.02\text{ m}$

(iii) By cosine rule, $ST^2 = SC^2 + TC^2 - 2SC.TC.\cos 30°$

$\therefore ST = 2982.91\text{ m}$

iv) In the diagram above, $\frac{\sin \angle CST}{TC} = \frac{\sin 30°}{ST}$

$\therefore \sin\angle CST = \frac{TC\sin 30°}{ST} \qquad \therefore \angle CST = 41.1°$

$\therefore$ Bearing is $58° + 41.1° = 99.1°$