NEWINGTON COLLEGE

IN FIDE SCIENTIAM

2015
Assessment 2 (HSC mini)

Year 12 Mathematics – Extension 1

General Instructions:

- Date of task - Monday 30th March (Wk 10B)
- Reading time – 5 mins
- Working time – 120 mins
- Weighting - 30%
- BOSTES approved calculators may be used.
- A table of standard integrals is provided at the back of the paper.
- Attempt all questions.
- Show all relevant mathematical reasoning and/or calculations.

Total marks – 70

Section I (10 marks)

- Answer questions 1 to 10 on the multiple choice answer sheet provided at the end of this paper.
- Allow about 15 minutes for this section.

Section II (60 marks)

- Answer questions 11 to 14 on the writing paper provided.
- **Start each question in a new writing booklet.**
- Each page must show the candidate's computer number.

Outcomes to be assessed:

HE1 Appreciates interrelationships between ideas drawn from different areas of mathematics.

HE4 Uses the relationship between functions, inverse functions and their derivatives.

HE6 Determines integrals by reduction to a standard form through a given substitution.

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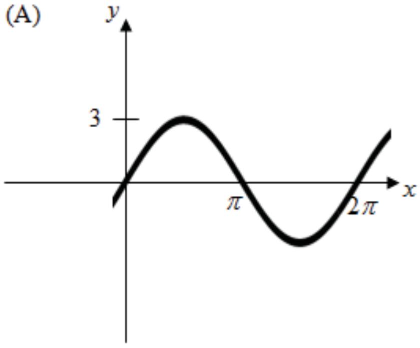
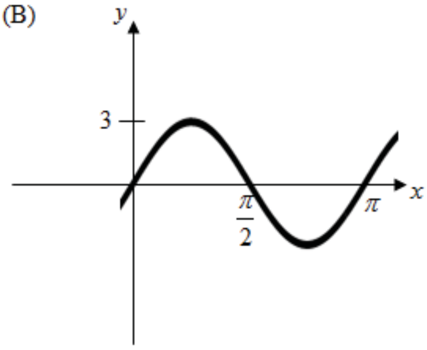
Section I **10 Marks**
Attempt Questions 1-10 on the multiple choice answer sheet.
Allow about 15 minutes for this section.

1) Which curve represents $y = 3\sin 2x$?

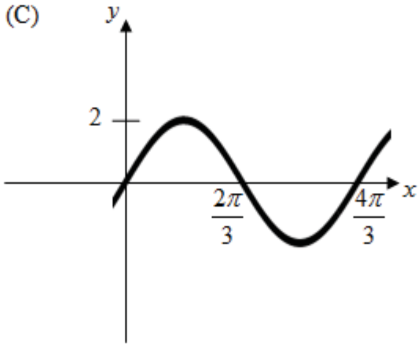
(A)



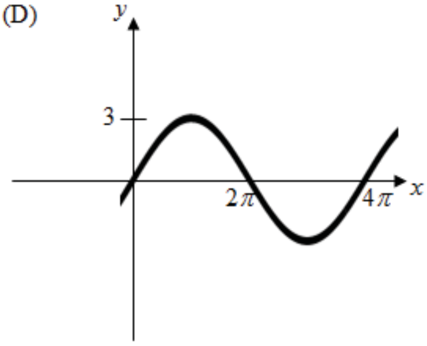
(B)



(C)



(D)

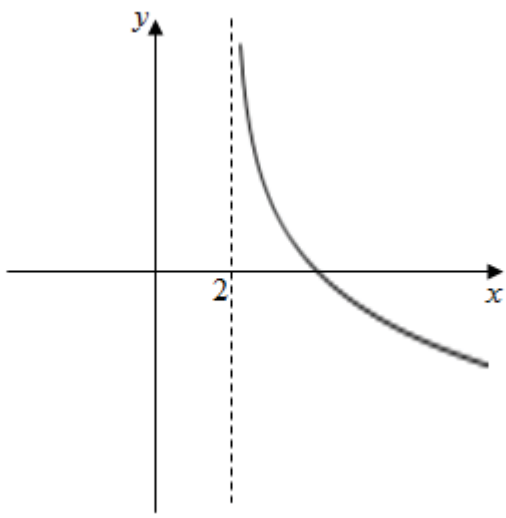


2) The acute angle *(to the nearest degree)* between two lines which have gradients of $\frac{1}{3}$ and $\frac{-2}{3}$ is:

(A) 1° (B) 23°

(C) 39° (D) 52°

3) What is a possible equation of the following curve?



(A) $y = \ln(x+2)$

(B) $y = \ln(x-2)$

(C) $y = \ln\left(\frac{1}{x+2}\right)$

(D) $y = \ln\left(\frac{1}{x-2}\right)$

4) Find $\int \cos x\, e^{\sin x} dx$

(A) $\sin x\, e^{\sin x} + C$

(B) $\cos^2 x\, e^{\sin x} + C$

(C) $e^{\sin x} + C$

(D) $\cos x\, e^{\sin x} + C$

5) Which function below only has an inverse function if the domain of $f(x)$ is restricted?

(A) $f(x) = \sin^{-1} x$

(B) $f(x) = \tan^{-1} x$

(C) $f(x) = \cos x$

(D) $f(x) = \log_e x$

6) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{3x}$ is equal to:

(A) 0

(B) 1

(C) $\dfrac{3}{2}$

(D) $\dfrac{2}{3}$

7) The expansion of $\cos(A + B)$ is equal to:

(A) $\sin A \cos B - \cos A \sin B$

(B) $\cos A \cos B - \sin A \sin B$

(C) $\sin A \cos B + \cos A \sin B$

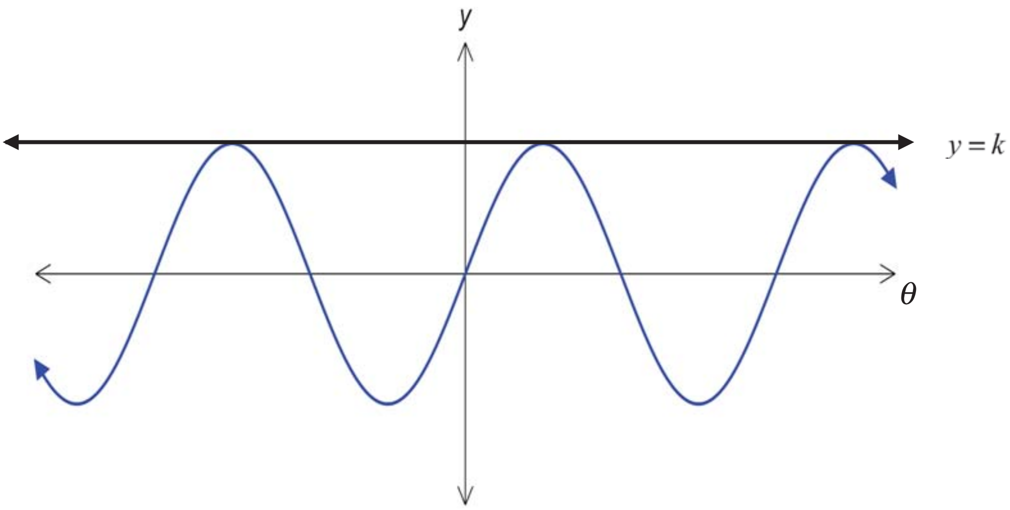
(D) $\cos A \cos B + \sin A \sin B$

8) The exact value of $\sin^{-1}\left(\cos\frac{2\pi}{3}\right)$ is:

(A) $\frac{\pi}{6}$ (B) $\frac{2\pi}{3}$

(C) $-\frac{\pi}{6}$ (D) $-\frac{2\pi}{3}$

9) The curve $y = 2\sin 2\theta$ and the horizontal line $y = k$ are drawn below.



The solutions of the equation $2\sin 2\theta - k = 0$ as shown on this diagram are:

(A) $\theta = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}$ (B) $\theta = -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

(C) $\theta = -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}$ (D) $\theta = -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$

10) $\int \frac{dx}{4+9x^2} =$

(A) $\frac{1}{3}\tan^{-1}\frac{2x}{3}$ (B) $\frac{1}{6}\tan^{-1}\frac{3x}{2}$

(C) $\frac{1}{6}\tan^{-1}\frac{2x}{3}$ (D) $\frac{1}{3}\tan^{-1}\frac{3x}{2}$

Section II
Attempt questions 11-14
Allow about 1 hour and 45 minutes for this section.

Question 11 (16 Marks)- Use a SEPARATE writing booklet.

a) Differentiate:

(i) $y=\cos(x^2-3)$ **1**

(ii) $f(x)=\tan^{-1}\left(\dfrac{5x}{4}\right)$ **2**

b) Prove $\dfrac{\cos x}{\sin x}+\dfrac{\sin x}{\cos x}=\dfrac{2}{\sin 2x}$ **3**

c) Find $\displaystyle\int\frac{\cos x}{\sin x}\,dx$ **1**

e) Find $\displaystyle\int\frac{1}{2}\sin 2x\sec x\,dx$ **2**

f) Find the exact value of $\sin 105^{\circ}$ **3**

g) If $\sin\theta=\dfrac{3}{5}$ and θ is acute, find the exact value(s) of $\tan\dfrac{\theta}{2}$ **4**

Question 12 (15 Marks)- Use a SEPARATE writing booklet.

a) Evaluate $\int_{1}^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$, leaving your answer in exact form. **3**

b) Evaluate $\int_{0}^{\frac{\pi}{2}} 3\sin 3\theta \ d\theta$ **2**

c) Find $\frac{d}{dx}\cos^{-1} x^2$ **2**

d) Find the general solution to $2\sin x = \sqrt{3}$,
expressing your answer in terms of π. **2**

e) Solve $1 + 2\cos^2 x = 5\sin x$, $0 \leq x \leq 2\pi$ **2**

f) (i) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $r\cos(\theta+\alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving r and α as exact values. **3**

(ii) Evaluate the minimum value of the expression $\sqrt{3}\cos\theta - \sin\theta$. **1**

Question 13 (15 Marks)- Use a SEPARATE writing booklet.

a) Given the function $f(x) = e^x + 2$

(i) Write an expression for the inverse function $f^{-1}(x)$ **2**

(ii) Write down the domain and range of $f^{-1}(x)$ **2**

b) Given the function $y = \dfrac{\ln x}{x}$

(i) Find any x and y intercepts **1**

(ii) Find any stationary points and determine their nature **3**

(iii) Clearly sketch the curve, indicating all the key features detailed above, as well as any asymptotes. **2**
(You do not have to show any points of inflexion)

c) Consider the function $f(x) = 3\cos^{-1}(x+1)$

(i) State the domain and range of this function. **2**

(ii) Draw a neat sketch of this function, showing this information. **1**

(iii) Find the equation of the tangent to this curve at the point $\left(-1, \dfrac{3\pi}{2}\right)$ **2**

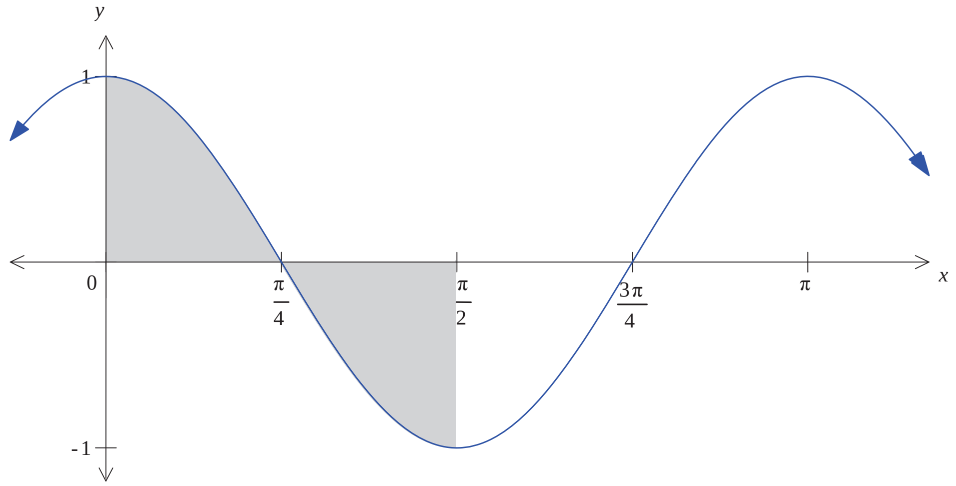
Question 14 (14 Marks)- Use a SEPARATE writing booklet.

a) Show that $\frac{d}{dx}(x.\sin^{-1}3x) = \sin^{-1}(3x) + \frac{3x}{\sqrt{1-9x^2}}$ **2**

Hence, or otherwise, find $\int \sin^{-1}3x\, dx$ **4**

(Hint: use the substitution $u = 1-9x^2$)

b) The area bounded by the curve $y = \cos 2x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis. **3**



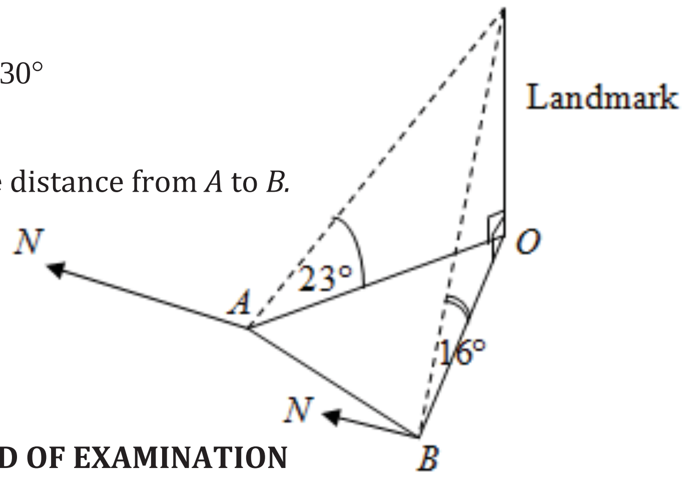
Find, in exact form, the volume of the solid of revolution formed.

c) The bearing of a vertical landmark from point A is $139°$ and the angle of elevation of the landmark from A is $23°$. From point B the same landmark has a bearing of $109°$ and an angle of elevation of $16°$. The landmark is 150m tall.

(i) Show that $AO = 150\cot 23°$ **1**

(ii) Show that $\angle AOB = 30°$ **2**

(iii) Hence, calculate the distance from A to B. **2**



END OF EXAMINATION

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Student Number :

SECTION I – Multiple Choice Answer Sheet

Instructions – 1. Tear off this page and write your student number in box above.
2. Colour in the circle corresponding to your correct answer.

Question 1	A ○	B ○	C ○	D ○
Question 2	A ○	B ○	C ○	D ○
Question 3	A ○	B ○	C ○	D ○
Question 4	A ○	B ○	C ○	D ○
Question 5	A ○	B ○	C ○	D ○
Question 6	A ○	B ○	C ○	D ○
Question 7	A ○	B ○	C ○	D ○
Question 8	A ○	B ○	C ○	D ○
Question 9	A ○	B ○	C ○	D ○
Question 10	A ○	B ○	C ○	D ○

Yr 12 Ext 1 - AP2 Mini - 2015

Section 1

1) Amp = 3 Period = $\frac{2\pi}{2} = 2$ (B)

2) $\tan\theta = \left[\dfrac{\frac{1}{3} - -\frac{2}{3}}{1 + \frac{1}{3} \times -\frac{2}{3}}\right]$

$\theta = \tan^{-1}\left[\dfrac{1}{\frac{7}{9}}\right] = \tan^{-1}\left[\frac{9}{7}\right]$

$\doteqdot 52^{0}$ (D)

3) Upside down shifted 2 to right.

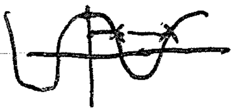
$\therefore y = -\ln(x-2)$

$y = \ln(x-2)^{-1}$

$y = \ln\left(\dfrac{1}{x-2}\right)$ (D)

4) In the form $\int f'(x)e^{f(x)}dx = e^{f(x)} + c$

(C)

5) 
more than 1 x for 1 y

$y = \cos x$ (C)

6) $\frac{2}{3} \lim_{x\to 0} \dfrac{\sin 2x}{2x} = \frac{2}{3}$ (D)

7) (B)

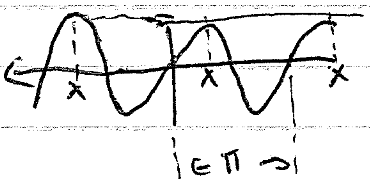
8) $\sin^{-1}\left(\cos\frac{2\pi}{3}\right) = \sin^{-1}\left(-\frac{1}{2}\right)$

$= -\sin^{-1}\left(\frac{1}{2}\right)$

$= -\frac{\pi}{6}$

(C)

9) Period $= \frac{2\pi}{2} = \pi$



$-\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$ (D)

10) $\int \dfrac{dx}{4+9x^2} = \frac{1}{9}\int \dfrac{dx}{\frac{4}{9}+x^2}$

$= \frac{1}{9} \times \dfrac{1}{\frac{2}{3}} \tan^{-1} \dfrac{x}{\frac{2}{3}} + c$

$= \frac{1}{6}\tan^{-1}\frac{3x}{2} + c$

(B)

Question 11

a) i) $y = \cos(x^2-3)$

$\frac{dy}{dx} = -2x\sin(x^2-3)$

ii) $f(x) = \tan^{-1}\frac{5x}{4}$

$f'(x) = \frac{5/4}{1+\left(\frac{5x}{4}\right)^2} = \frac{5}{4+\frac{25x^2}{4}}$

$= \frac{20}{16+25x^2}$

b) $LHS = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$

$= \frac{\cos^2 x + \sin^2 x}{\sin x\cos x}$

$= \frac{1}{\sin x\cos x}$

$= \frac{2}{2\sin x\cos x}$

$= \frac{2}{\sin 2x} = RHS$

c) $\int \frac{\cos x}{\sin x}\,dx = \ln \sin x + c$

d) $\int \frac{1}{2}\sin 2x \sec x\,dx = \int \frac{1}{2}\frac{2\sin x\cos x}{\cos x}\,dx$

$= \int \sin x\,dx$

$= -\cos x + c$

e) $\sin(105) = \sin(45+60)$

$= \sin 60\cos 45 + \cos 60\sin 45$

$= \frac{\sqrt{3}}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

f) $\sin\theta = \frac{3}{5}$

$\therefore \frac{3}{5} = \frac{2t}{1+t^2}$

$3+3t^2 = 10t$

$3t^2-10t+3=0$

$(3t-1)(t-3)=0$

$t=\frac{1}{3}$ or $t=3$

$\tan\frac{\theta}{2} = \frac{1}{3}$ or $\tan\frac{\theta}{2} = 3$

θ is acute $\therefore \tan\frac{\theta}{2} = \frac{1}{3}$

2/6

Question 12

a) $\int_1^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right)\Big|_1^{\sqrt{3}}$

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

b) $\int_0^{\frac{\pi}{2}} 3\sin 3\theta \, d\theta = \left[-\cos 3\theta\right]_0^{\frac{\pi}{2}}$

$= -\cos\frac{3\pi}{2} - -\cos 0$

$= 1$

c) $\frac{d}{dx}\cos^{-1}x^2$

$u = x^2$

$\frac{du}{dx} = 2x$

$y = \cos^{-1}u$

$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^2}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{-1}{\sqrt{1-u^2}} \times 2x$

$= \frac{-2x}{\sqrt{1-x^4}}$

d) $\sin x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{3}, \pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}$

$x = n\pi + (-1)^n \frac{\pi}{3}$

e) $1 + 2\cos^2 x = 5\sin x$

$2\cos^2 x - 5\sin x + 1 = 0$

$2(1 - \sin^2 x) - 5\sin x + 1 = 0$

$2 - 2\sin^2 x - 5\sin x + 1 = 0$

$2\sin^2 x + 5\sin x - 3 = 0$

$(2\sin x - 1)(\sin x + 3) = 0$

$\sin x = \frac{1}{2}$ $\qquad$ $\sin x = -3$

NO SOLN

$\Downarrow$

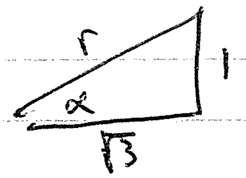
$x = \frac{\pi}{6}, \frac{5\pi}{6}$

f) $\sqrt{3}\cos\theta - \sin\theta = r\cos(\theta + \alpha)$

$= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha$

$\sqrt{3} = r\cos\alpha \qquad 1 = r\sin\alpha$

$\cos\alpha = \frac{\sqrt{3}}{r} \qquad \sin\alpha = \frac{1}{r}$



$r = 2$

$\alpha = \frac{\pi}{6}$

$\therefore \sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta + \frac{\pi}{6}\right)$

ii) min value $= -2$

3/6

Question 13

) f: $y = e^x + 2$

f^{-1}: $x = e^y + 2$

$e^y = x - 2$

$y = \log_e(x-2)$

i) D: $x > 2$

R: all real y

) $y = \frac{\ln x}{x}$

(i) x int: $y=0$

$0 = \frac{\ln x}{x}$

$\ln x = 0 \quad \therefore x = 1$

no y intercept

(ii) $y' = \frac{\frac{1}{x} \times x - 1 \times \ln x}{x^2}$

$= \frac{1 - \ln x}{x^2}$

t Pts: $y' = 0$

$0 = 1 - \ln x$

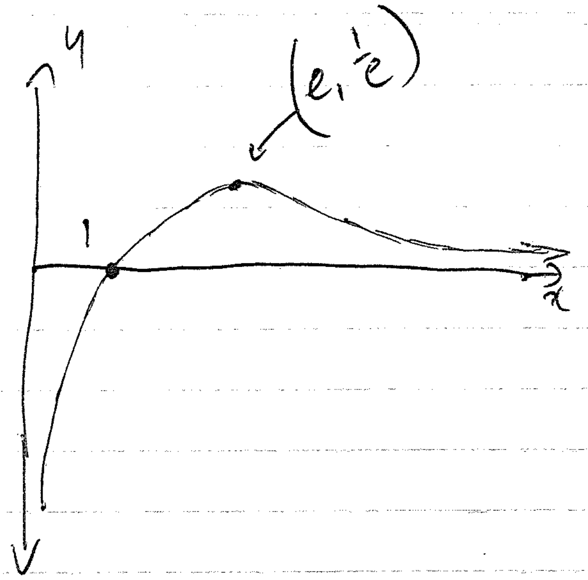
$\ln x = 1 \quad \therefore x = e$

EST

x	2	e	3
y'	$\frac{1-\ln 2}{4}$	0	$\frac{1-\ln 3}{9}$
	/	—	\

$\therefore$ max at $\left(e, \frac{1}{e}\right)$

iii)



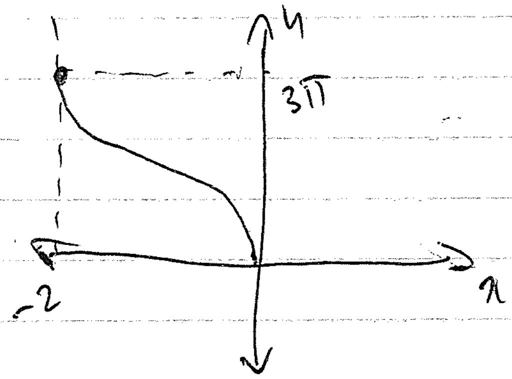
c) $f(x) = 3\cos^{-1}(x+1)$

i) D: $-1 \leq x+1 \leq 1$

$-2 \leq x \leq 0$

R: $0 \leq y \leq 3\pi$

ii)



iii) $f(x) = 3\cos^{-1}(x+1)$

$f'(x) = \frac{-1}{\sqrt{1-(x+1)^2}}$

$f'(-1) = -\frac{3}{1} = -3$

$y - \frac{3\pi}{2} = -3(x+1)$

$3x + y - \frac{3\pi}{2} + 3 = 0$ //

4/6

Question 14

a) $\frac{d}{dx}\left(x\sin^{-1}3x\right)$

$= \frac{x}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} + \sin^{-1}3x$

$= \frac{3x}{\sqrt{1-9x^2}} + \sin^{-1}3x$

ii) $\int \frac{3x}{\sqrt{1-9x^2}} + \sin^{-1}3x \, dx = x\sin^{-1}3x$

$\therefore \int \sin^{-1}3x \, dx = x\sin^{-1}3x - \int \frac{3x}{\sqrt{1-9x^2}} \, dx$ ①

$u = 1-9x^2$

$\frac{du}{dx} = -18x, \quad dx = \frac{du}{-18x}$

① $\Rightarrow \int \frac{3x}{\sqrt{u}} \cdot \frac{du}{-18x}$

$= -\frac{1}{6}\int \frac{du}{\sqrt{u}}$

$= -\frac{1}{6} \cdot 2\sqrt{u}$

$= x\sin^{-1}3x + \frac{1}{3}\sqrt{u}$

$= x\sin^{-1}3x + \frac{1}{3}\sqrt{1-9x^2} + C$

//

b) $V = \pi\int_0^{\frac{\pi}{2}} y^2 \, dx \qquad y^2 = \cos^2 2x$

$= \pi\int_0^{\frac{\pi}{2}} \cos^2 2x \, dx$

$\cos 2x = 2\cos^2 x - 1$

$\therefore \cos^2 x = \frac{1}{2}(1+\cos 2x)$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} 1 + \cos 4x \, dx$

$= \frac{\pi}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$

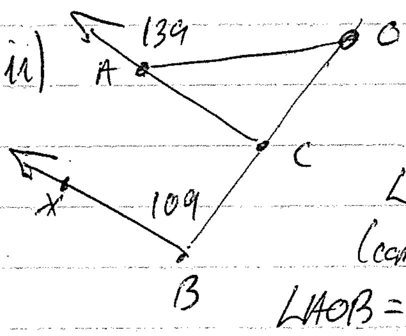
$= \frac{\pi}{2}\left[\left(\frac{\pi}{2} + \frac{1}{4}\sin\frac{4\pi}{2}\right) - \left(0 + \frac{1}{4}\sin 0\right)\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{2} + 0\right] = \frac{\pi^2}{4}$

c) i) $\tan 23 = \frac{150}{AO}$

$\therefore AO = 150\cot 23$

ii)



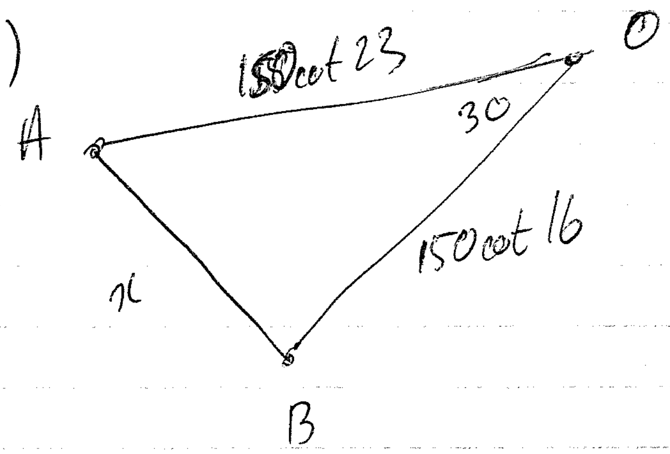
$\angle ACO = 109°$ (corresponding to $\angle CBX$)

$\angle AOB = 139 - 109$ (exterior angle)

$\angle AOB = 30°$

5/6

ii)



$$x^2 = (150\cot 23)^2 + (150\cot 16)^2$$
$$- 2 \cdot 150^2 \cot 13 \cdot \cot 16 \cdot \cos 30$$

$$x = 150\sqrt{\frac{1}{\tan^2 23} + \frac{1}{\tan^2 16} - \frac{2\cos 30}{\tan 23 \tan 16}}$$

$x = 279.896\text{m}$ //

6/6