Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2010 HSC ASSESSMENT TASK 2

Mathematics

Extension 1

General Instructions

- Working time – 60 minutes
- Write on the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Ireland
- O Mr Lowe
- O Mr Rezcallah
- O Mr Barrett
- O Mr Trenwith
- O Mr Weiss

Student Number:________________________

(To be used by the exam markers only.)

Question No	1	2	3	4	5	Total	Total %
Mark	/17	/13	/12	/10	/14	/66	/100

Question 1 (17 marks) **Marks**

(a) Evaluate $\frac{\pi}{e^3}$ correct to 3 significant figures. **1**

(b) Find $\frac{dy}{dx}$ for the following:

(i) $y = x^2 e^{3x}$ **2**

(ii) $y = \sqrt{\ln 2x}$ **2**

(iii) $y = 3^x$ **1**

(c) (i) Find $\int \frac{x^2}{x^3 + 3}\, dx$ **2**

(ii) Evaluate $\int_0^1 x^2 e^{3x}\, dx$ **2**

(d) If $y = \ln\left(\frac{e^x + 1}{e^x - 1}\right)$, find $\frac{dy}{dx}$ as a fraction in simplest form. **3**

(e) If $f(x) = \ln(x-3)(5-x)$, what is the domain of $f(x)$? **2**

(f) Solve the equation $2\ln(x-1) = \ln(4x-7)$. **2**

Question 2 (13 marks)

(a) Use Mathematical Induction to prove that $4^n - 1$ is divisible by 3 for all $n \geq 1$. **5**

(b) (i) Show that the equation $x^3 + 2x - 4 = 0$ has a root α in the interval $1 < \alpha < 2$. **1**

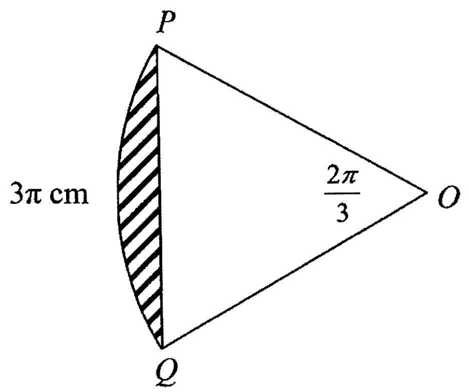
(ii) Show that this equation has **exactly** one root in this interval. (Hint: You might want to consider the derivative.) **2**

(iii) Use Newton's method to show that if $x_1 = a$ is an approximation to α, then the next approximation is $x_2 = \frac{2a^3 + 4}{3a^2 + 2}$. **3**

(iv) Use Newton's method once with an initial approximation of $x_1 = 2$ to find a better approximation for α. (Give your answer correct to 2 decimal places.) **2**

Question 3 (12 marks)

(a)



In the diagram above, PQ is the arc of a circle, centre O.
The length of the arc is 3π cm, and angle POQ is $\dfrac{2\pi}{3}$.

(i) Find the radius of the circle. **2**

(ii) Find the area of the sector OPQ. **2**

(iii) Find the area of the shaded region. **2**

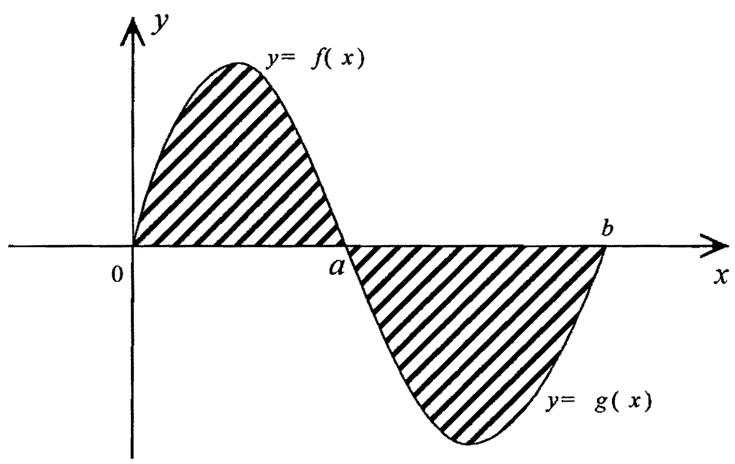
(b) (i) State the period and amplitude of the function $y = 3 - \cos 2x$. **2**

(ii) Sketch the curve $y = 3 - \cos 2x$ for $0 \leq x \leq 2\pi$. **3**

(c) Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x}$. **1**

Question 4 (10 marks)

(a) **2**



Write an expression involving integrals for the area of the shaded region.

(b) Find the area enclosed between the hyperbola $y = \frac{4}{x}$ and the line $y = 5 - x$. **4**

(c) (i) Differentiate $y = \ln(x^2)$. **1**

(ii) Hence evaluate $\int_e^{e^2} \frac{dx}{x\ln(x^2)}$. **3**

Question 5 (14 marks)

(a) (i) Find the stationary point on the curve $y = e^x + xe^x$, and determine its nature. **3**

(ii) Find any points of inflexion. **3**

(iii) Find where the above curve cuts the coordinate axes. **2**

(iv) Sketch the curve $y = e^x + xe^x$, . **2**

(b) Find the volume of the solid generated when the region bounded by the curve $y = \ln x$, the x-axis, and the lines $x = 1$ and $x = 2$ is rotated about the y-axis. **4**

Q1) a) 0.156 1

b) i) $y = x^2 e^{x^3}$

$\frac{dy}{dx} = 2x e^{x^3} + x^2 . 3x^2 e^{x^3}$ 2

$= x e^{x^3}(2 + 3x^3)$

ii) $y = (\ln 2x)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(\ln 2x)^{-1/2} . \frac{2}{2x}$ 2

$= \frac{1}{2x(\ln 2x)^{1/2}}$

iii) $y = 3^x$

$\ln y = x \ln 3$

$\frac{1}{y}\frac{dy}{dx} = \ln 3$

$\frac{dy}{dx} = \ln 3 . 3^x$ 1

c) i) $\frac{1}{3}\int \frac{3x^2}{x^3+3}\,dx$

$= \frac{1}{3}\ln(x^3+3) + c$ -1 2

ii) $\frac{1}{3}\int_0^1 3x^2 e^{x^3}\,dx$

$= \frac{1}{3} e^{x^3}\Big]_0^1$ 1

$= \frac{1}{3}e - \frac{1}{3}$ 1

d) $y = \ln(e^x+1) - \ln(e^x-1)$

$\frac{dy}{dx} = \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1}$ 2

$= \frac{e^{2x} - e^x - e^{2x} - e^x}{(e^x+1)(e^x-1)}$

$= \frac{-2e^x}{(e^x+1)(e^x-1)}$ 1

e) $f(x) = \ln(x-3)(5-x)$

$(x-3)(5-x) > 0$ 1

D: $3 < x < 5$ 1

f) $\ln(x-1)^2 = \ln(4x-7)$

$(x-1)^2 = 4x-7$

$x^2 - 2x + 1 = 4x - 7$ 1

$x^2 - 6x + 8 = 0$

$(x-4)(x-2) = 0$

$\therefore x = 2$ or 4 1

Question 2 a) Two methods shown:

When $n=1$:

$4^1 - 1 = 3$

$\therefore$ True when $n=1$ ✓

Assume true for $n=k$

$4^k - 1 = 3N$ where N is an integer

Prove true for $n=k+1$

$4^{k+1} - 1 = 4.4^k - 1$

$= 4(4^k - 1) + 3$

$= 4(3N) + 3$ ✓

$= 3(4N+1)$

$\therefore$ True for $n=k+1$ if true for $n=k$. ✓

Since true for $n=1$, also true for $n=2$, and also for $n=3$ and so on. ✓

$\therefore 4^n - 1$ is divisible by 3 for all $n \geq 1$.

Qn 2

a) Let $S(n)$ be statement that $4^n - 1$ is divisible by 3 for $n \geq$

For $n=1$:

$LHS = 4^1 - 1 = 3$

i.e. $S(1)$ is true ✓

Assume $S(k)$ is true

i.e. $4^k - 1 = 3M$ where M is an integer

$\therefore 4^k = 3M+1$

RTP $S(k+1)$ is true

i.e. $4^{k+1} - 1$ is divisible by 3 ✓

$4^{k+1} - 1 = 4^{k+1} - 1$

$= 4 \cdot 4^k - 1$

$= 4(3M+1) - 1$ ✓

$= 12M + 3$

$= 3(4M+1)$ which is divisible by 3 since $4M+1$ is an integer

$\therefore$ If $S(k)$ is true, then $S(k+1)$ is true ✓

Since $S(1)$ is true, then $S(2)$ is true and $S(3)$ is true and so on

$\therefore$ By MI, $S(n)$ is true for all $n \geq 1$ ✓

b) i/ $x^3 + 2x - 4 = 0$

Let $f(x) = x^3 + 2x - 4$

$f(1) = -1 < 0$

$f(2) = 8 > 0$ ✓ for showing opposite signs

$\therefore$ root between $x=1$ and 2

ii/ $f'(x) = 3x^2 + 2$

> 0 for all x ✓

$\therefore$ curve ~~concave up for all x~~ is increasing for all x ✓

$\therefore$ crosses x-axis once

(Alt: show no solution to $3x^2 + 2 = 0$; then ~~show~~ state this means no stationary/turning points).

$x_2 = a - \dfrac{a^3 + 2a - 4}{3a^2 + 2}$ ✓✓

$= \dfrac{3a^3 + 2a - a^3 - 2a + 4}{3a^2 + 2}$

$= \dfrac{2a^3 + 4}{3a^2 + 2}$ ✓

$x_2 = \dfrac{2(2)^3 + 4}{3(2)^2 + 2}$ ✓

$= 1.43$ (to 2 D) ✓

Q3)

a) i/ $l = r\theta$

$3\pi = r\,\frac{2\pi}{3}$ 1

$\therefore r = 3\pi \times \frac{3}{2\pi}$

$= \frac{9}{2}$ cm 1

ii/ $A = \frac{1}{2} r^2 \theta$

$= \frac{1}{\cancel{2}} \times \frac{81}{4} \times \frac{\cancel{2}\pi}{\cancel{3}}$ 1

$= \frac{27\pi}{4}$ cm^2 or 21.21 1

iii/ $A = \frac{27\pi}{4} - \frac{1}{2} \times \frac{9}{2} \times \frac{9}{2} \sin\frac{2\pi}{3}$ 1

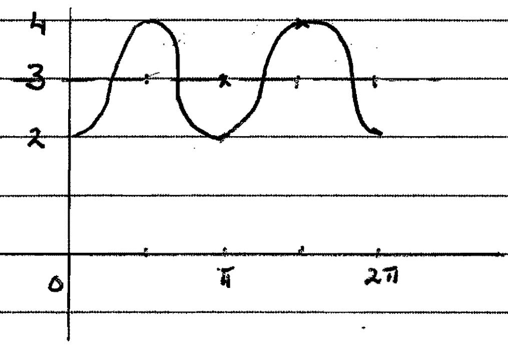
$= \frac{27\pi}{4} - \frac{81}{8} \cdot \frac{\sqrt{3}}{2}$

$= \frac{27\pi}{4} - \frac{81\sqrt{3}}{16}$ cm^2 or $\frac{108\pi - 81\sqrt{3}}{16}$ or 12.437 1

b) i/ amp = 1 1

period = π 1

ii/



3

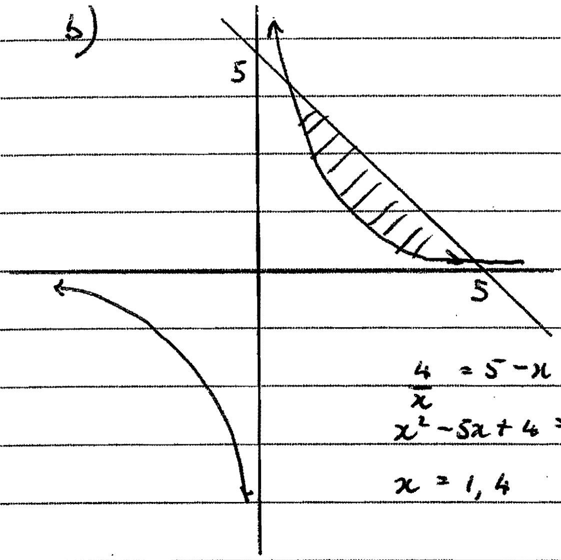
c) $2 \lim_{x \to 0} \frac{\sin 2x}{2x}$

$= 2 \times 1$

$= 2$ 1

Q4) a) $A = \int_0^a f(x)\,dx + \left| \int_a^5 g(x)\,dx \right|$ or $A = \int_0^a f(x)\,dx - \int_a^5 g(x)\,dx$. 2

b)



$\frac{4}{x} = 5 - x$

$x^2 - 5x + 4 = 0$

$x = 1, 4$ 1

$A = \int_1^4 \left(5 - x - \frac{4}{x}\right) dx$ 1

$= 5x - \frac{x^2}{2} - 4\ln x \Big]_1^4$

$= 20 - 8 - 4\ln 4 - 5 + \frac{1}{2}$ 1

$= \frac{15}{2} - 4\ln 4 \quad u^2$ 1

c) i/ $y = \ln(x^2)$

$\frac{dy}{dx} = \frac{2}{x}$ 1

ii/ $\frac{1}{2}\int_e^{e^2} \frac{\frac{2}{x}}{(\ln x^2)}\,dx$

1

$= \frac{1}{2} \ln\left[\ln x^2\right]_e^{e^2} = \frac{1}{2}\left[\ln(\ln e^4) - \ln(\ln e^2)\right] = \frac{1}{2}(\ln 4 - \ln 2)$

1

$= \frac{1}{2}\ln 2$ 1

The hence has to be used. Students who work by substitution and get the correct answer, lose 1 or 2 marks.

Q5)

a) i/ $y = e^x + xe^x$

$$\frac{dy}{dx} = e^x + e^x + xe^x$$
$$= 2e^x + xe^x$$
$$= e^x(2+x)$$

$\frac{dy}{dx} = 0$ when $x = -2$ 1

$$\frac{d^2y}{dx^2} = 2e^x + e^x + xe^x$$
$$= 3e^x + xe^x$$ 1

when $x = -2$

$$\frac{d^2y}{dx^2} = \frac{3}{e^2} - \frac{2}{e^2} = \frac{1}{e^2}$$
$$> 0$$

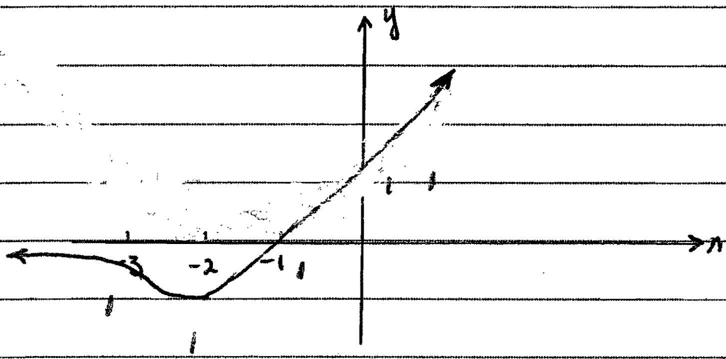
$\therefore$ min at $\left(-2, \frac{-1}{e^2}\right)$ 1

ii/ $\frac{d^2y}{dx^2} = 0$ when $x = -3$ 1

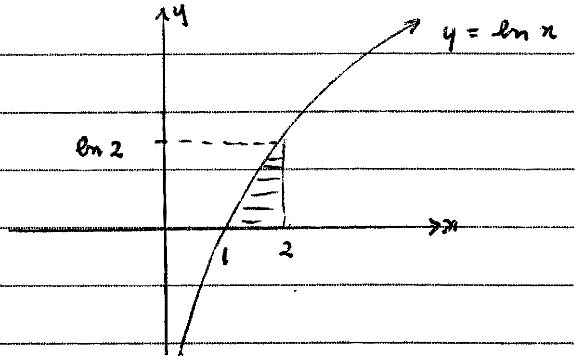
x	-4	-3	-2
$\frac{d^2y}{dx^2}$	<0	$=0$	>0

1

$\therefore$ infl at $\left(-3, \frac{-2}{e^3}\right)$



b)



$$V = \pi \int x^2\, dy$$
$$= 4\pi \ln 2 - \pi \int_0^{\ln 2} e^{2y}\, dy$$
$$= 4\pi \ln 2 - \pi \left[\tfrac{1}{2} e^{2y}\right]_0^{\ln 2}$$
$$= 4\pi \ln 2 - \frac{\pi}{2}\left[4 - 1\right]$$
$$= \left(4\pi \ln 2 - \frac{3\pi}{2}\right) u^3$$