Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2013 HSC ASSESSMENT TASK 2

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 55 minutes
- Write on the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Lucas
- O Mr Berry
- O Mr Lin
- O Mr Fletcher
- O Mr Lam
- O Ms Ziaziaris

Student Name/Number: ____________________________

(To be used by the exam markers only.)

Question No	1-5	6	7	8	9	10	11	Total	Total %
Mark	$\frac{}{5}$	$\frac{}{10}$	$\frac{}{8}$	$\frac{}{6}$	$\frac{}{10}$	$\frac{}{3}$	$\frac{}{7}$	$\frac{}{49}$	$\frac{}{100}$

SECTION A – MULTIPLE CHOICE ***(5 Marks)***

1. If $y = e^{x^2}$ which of the following is an expression for $\frac{d^2y}{dx^2}$

(A) $2xe^{x^2}$
(B) $2e^{x^2}$
(C) $4xe^{x^2}$
(D) $(4x^2 + 2)e^{x^2}$

2. If $e^{x+2} = 4$, the exact value of x is:

(A) 2
(B) $e^4 + 2$
(C) $log_e 2$
(D) $log_e 4 - 2$

3. The polynomial $P(x) = ax^3 + bx^2 + cx + d$ has zeros at: $-2, -1, 1$ and 2. What is the value of b

(A) -2
(B) -1
(C) 0
(D) 2

4. The function $y = x^3 + 2x^2 + 3x + 4$ has a root between -1 and -2. By halving the interval a better approximation can be determined to be between:

(A) -1 and -1.25
(B) -1.25 and -1.5
(C) -1.5 and -1.75
(D) -1.75 and -2

5. The gradient of the normal to the curve $y = log_e x^2$ at $x = 2$ is:

(A) -1
(B) 0
(C) 1
(D) 4

SECTION B – EXTENDED RESPONSE

Question 6 (10 Marks)

(a) Differentiate

(i) $log_{10}x$ **1**

(ii) x^2e^x **2**

(b) Find the primitive of:

(i) $2e^x + 1$ **1**

(ii) $\frac{x}{2x^2+1}$ **2**

(iii) $\frac{2x^2+1}{x}$ **2**

(c) Evaluate $\int_1^3 e^{2x}dx$ correct to two decimal places: **2**

Question 7 *(8 Marks)*

(a) (i) Factorise $x^3 - x^2 - 8x + 12$ completely **3**

(ii) Hence sketch $P(x) = x^3 - x^2 - 8x + 12$ **2**

(you are not required to use calculus)

(b) The root of $f(x) = 0.4x - e^{-x^2}$ is near $x = 1$. Use Newton's Method once to find a 2 decimal place approximation to this root **3**

Question 8 *(6 Marks)*

(a) If α, β *and* γ are the roots of the equation $3x^3 - 5x^2 + 2x - 3 = 0$.

Find the values of

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta + \beta\gamma + \alpha\gamma$ **1**

(iii) $\alpha^2 + \beta^2 + \gamma^2$ **1**

(b) The polynomial $P(x) = x^3 + ax^2 - bx + 6$ has $(x - 3)$ as a factor.

When $P(x)$ is divided by $(x + 1)$ the remainder is 8.

Find the values of a and b. **3**

Question 9 ***(10 Marks)***

(a) Find the volume when $y = \frac{2}{\sqrt{x}}$ is rotated around the x–axis between $x = 3$ and $x = 5$ **3**

(b) (i) Draw a neat sketch of $y = log_e(x-1)$ showing all relevant features **2**

(ii) Use 2 applications (i.e. five function values) of Simpson's rule to estimate the area enclosed by $y = log_e(x-1)$, $x = 6$ and the x-axis, in the first quadrant correct to two decimal places. **3**

(iii) Hence, or otherwise, approximate the area enclosed by the lines $y = log_e(x-1)$, $y = log_e 5$ and the coordinate axes. **2**

Question 10 ***(3 Marks)***

(a) Prove by mathematical induction that $7^n - 6n - 1$ is divisible by 36 for all positive integers $n \geq 2$. **3**

Question 11 ***(7 Marks)***

(a) Solve for x: $log_2 x + log_2(x+7) = 3$ **3**

(b) If $y = \frac{log_e x}{x}$

(i) Show that $\frac{dy}{dx} = \frac{1 - log_e x}{x^2}$ **1**

(ii) Hence or otherwise show that:

$$\int_e^{e^2} \frac{1 - \log_e x}{x \log_e x} dx = (\log_e 2) - 1$$ **3**

End of Examination

MULTIPLE CHOICE

1. D
2. D
3. C
4. C
5. A

Question 6

(a) (i) $\frac{d}{dx}\log_{10} x = \frac{1}{x\log_e 10}$	✓ correct answer
(a) (ii) $y = x^2 e^x$ let $u = x^2$ $\therefore u' = 2x$ let $v = e^x$ $\therefore v' = e^x$ $y' = u'v + v'u$ $= (2x)(e^x) + (e^x)(x^2)$ $= xe^x(x+2)$	✓ product rule ✓ answer
(b) (i) $\int (2e^x + 1)\,dx = 2e^x + x + c$	✓ correct answer
(b) (ii) $\int \frac{x}{2x^2+1}\,dx$ $= \frac{1}{4}\int \left(\frac{4x}{2x^2+1}\right) dx$ $= \frac{1}{4}\log_e(2x^2+1) + c$	✓ any log as the primitive ✓ correct answer
(b) (iii) $\int \frac{2x^2+1}{x}\,dx$ $= \int \left(2x + \frac{1}{x}\right) dx$ $= x^2 + \log_e x + c$	✓ breaking up fraction ✓ correct answer

(c) $\int_1^3 e^{2x}\,dx$

$= \left[\frac{1}{2} e^{2x}\right]_1^3$ ✓ correct primitive

$= \frac{1}{2}(e^6 - e^2)$

$= 198.02$ (2 d.p.) ✓ correct answer.

Question 7

(a)(i) let $P(x) = x^3 - x^2 - 8x + 12$

$$P(2) = 2^3 - 2^2 - 8\times 2 + 12$$
$$= 0$$

$\therefore (x-2)$ is a factor

✓ finding a factor

$$\begin{array}{r} x^2 + x - 6 \\ x-2 \,\big)\, \overline{x^3 - x^2 - 8x + 12} \\ \underline{x^3 - 2x^2} \\ x^2 - 8x \\ \underline{x^2 - 2x} \\ -6x + 12 \\ \underline{-6x + 12} \end{array}$$

✓ polynomial division

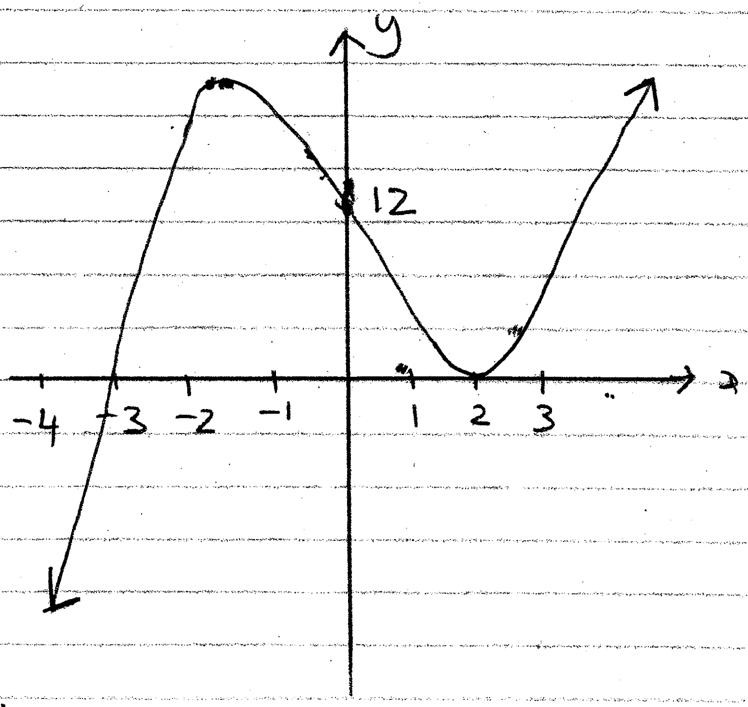
$\therefore P(x) = (x-2)(x^2 + x - 6)$

$\therefore P(x) = (x-2)(x+3)(x-2)$

✓ factorised expression

$P(x) = (x-2)^2(x+3)$

(a)(ii)



✓ correct single & double root

✓ shape

(b) $f(x) = 0.4x - e^{-x^2}$

$\therefore f'(x) = 0.4 + 2x e^{-x^2}$ ✓ correct derivative

$a_0 = 1$

$a_1 = a_0 - \dfrac{f(a_0)}{f'(a_0)}$

$= 1 - \dfrac{0.4 - e^{-1}}{0.4 + 2e^{-1}}$ ✓ subbing $a_0 = 1$ into formula

$\therefore a_1 = 0.97$ (2 d.p.) ✓ final answer

Question 8

(a)(i) $\alpha + \beta + \gamma = -b/a$

$= 5/3$ ✓ correct answer

(a)(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = c/a$

$= 2/3$ ✓ correct answer

(a)(iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (5/3)^2 - 2(2/3)$

$= 13/9$ ✓ correct answer

(b) $P(3) = 0$

$\therefore 3^3 + a \times 3^2 - 3b + 6 = 0$

$33 + 9a - 3b = 0$

$9a - 3b = -33 \quad \ldots \; (1)$ ✓ simultaneous equations

$P(-1) = 8$

$\therefore (-1)^3 + a(-1)^2 + b + 6 = 8$

$-1 + a + b + 6 = 8$

$a + b = 3$

$\therefore b = 3 - a \quad \ldots \; (2)$

sub (2) $\rightarrow$ (1)

$9a - 3(3-a) = -33$	
$9a - 9 + 3a = -33$	
$12a = -24$	
$a = -2$	✓ solution to a
sub into ②	
$b = 3 - a$	
$= 3 - (-2)$	
$\therefore b = 5$	✓ solution to b

Question 9

(a) $V = \pi \int_a^b y^2 \, dx$

$= \pi \int_3^5 \left(\frac{2}{\sqrt{x}}\right)^2 dx$

$= \pi \int_3^5 4/x \, dx$ ✓ subbing into formula

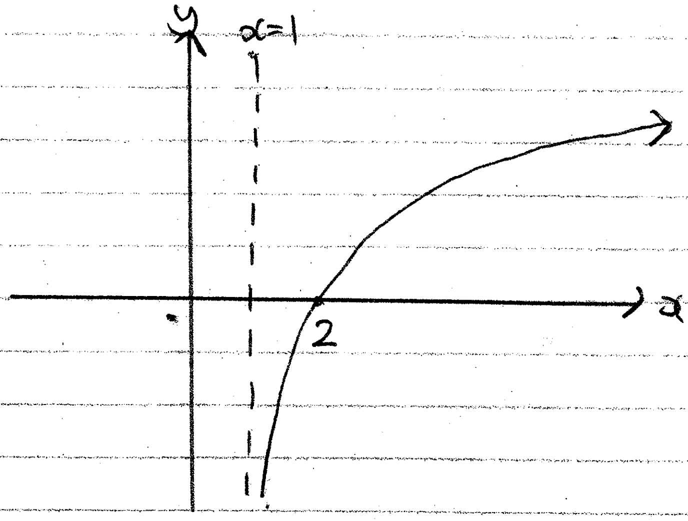
$= 4\pi \int_3^5 1/x \, dx$

$= 4\pi [\log_e x]_3^5$ ✓ correct integral

$= 4\pi [\log_e 5 - \log_e 3]$

$= 4\pi \log_e (5/3)$ ✓ final answer

(b)(i)



✓ shape

✓ asymptote AND x-intercept

(b)(ii)

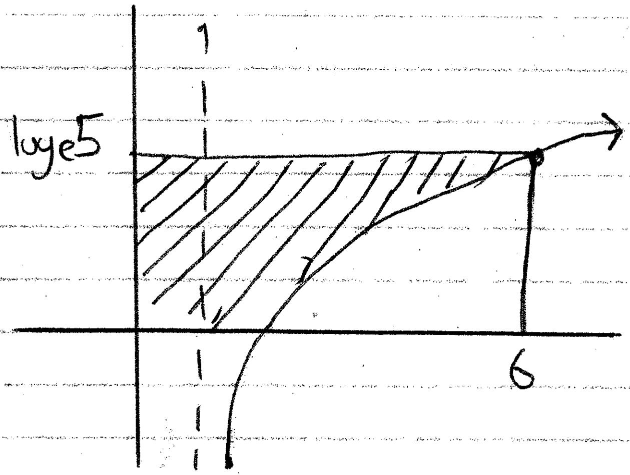
y_0	y_1	y_2	y_3	y_4
$\log_e 1$	$\log 2$	$\log 3$	$\log 4$	$\log 5$

$A \doteqdot \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 \ldots) + 2(y_2 + y_4 \ldots)]$ ✓ correct formula

$= \frac{1}{3}[(\log 1 + \log 5) + 4(\log 2 + \log 4) + 2(\log 3)]$ ✓ correct values

$= 4.04$ (2 d.p.) ✓ final answer

(b)(iii)



$A \doteqdot (6 \times \log_e 5) - 4.04$

$= 5.615$ (3 d.p.)

✓ area of rectangle

✓ final answer (any rounding is ok)

Question 10

1./ when $n=2$

$7^2 - 6\times 2 - 1 = 36$

which is divisible by 36

✓ proving for base case

2./ assume true for $n=k$ where k is an integer

i.e. $7^k - 6k - 1 = 36m$ where m is an integer

3./ for $n=k+1$

$7^{k+1} - 6(k+1) - 1$

$= 7\cdot 7^k - 6k - 6 - 1$

$= 7\times 7^k - 6k - 7$

$= 7\times 7^k - 42k - 7 - 6k + 42k$

$= 7(7^k - 6k - 1) + 36k$

$= 7\times 36m + 36k$, from assumption

✓ substitution from assumption

$= 36(7m + k)$

which is divisible by 36 since m & k are both integers

✓ correctly showing divisibility

4./ therefore the proposition is true by the principle of mathematical induction

Question 11

(a) $\log_2 x + \log_2(x+7) = 3$

$\log_2(x^2+7x) = 3$ ✓ combining into 1 log

$\therefore x^2 + 7x = 2^3$ ✓ re-arranging to get exponential

$x^2 + 7x = 8$

$x^2 + 7x - 8 = 0$

$(x+8)(x-1) = 0$

$\therefore x = -8$ or $x = 1$

$\therefore x = 1$ as $x > 0$ ✓ correct final answer must have only 1 final solution

(b)(i) $y = \dfrac{\log_e x}{x}$

let $u = \log_e x$ & $v = x$

$\therefore u' = 1/x$ & $v' = 1$

$\dfrac{dy}{dx} = \dfrac{u'v - v'u}{v^2}$

$= \dfrac{(1/x)(x) - (1)\log_e x}{x^2}$

$= \dfrac{1 - \log_e x}{x^2}$ "as req'd" ✓ all necessary steps shown (b)(ii) $\displaystyle\int_e^{e^2} \frac{1-\log_e x}{x\log_e x}\,dx$

$$= \int_e^{e^2} \frac{\left(\frac{1-\log_e x}{x^2}\right)}{\left(\frac{x\log_e x}{x^2}\right)}\,dx$$

✓ dividing by x^2

$$= \int_e^{e^2} \frac{\left(\frac{1-\log_e x}{x^2}\right)}{\left(\frac{\log_e x}{x}\right)}\,dx$$

$$= \left[\log_e\left(\frac{\log_e x}{x}\right)\right]_e^{e^2} \quad \text{(from (i))}$$

✓ correct integral

$$= \log_e\left(\frac{\log_e e^2}{e^2}\right) - \log_e\left(\frac{\log_e e}{e}\right)$$

$$= \log_e\left(\frac{2}{e^2}\right) - \log_e\left(\frac{1}{e}\right)$$

$$= \log_e 2 - \log_e e^2 - \log_e 1 + \log_e e$$

$$= \log_e 2 - 2 + 0 + 1$$

$$= (\log_e 2) - 1 \text{, as required}$$

✓ successfully getting to final answer.