Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2015 HSC ASSESSMENT TASK 2

Mathematics
Extension 1

General Instructions

- Working time – 55 minutes (+ 5 minutes reading time)
- Write on the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

Class Teacher:
(Please tick or highlight)

- O Mr Berry
- O Ms Ziaziaris
- O Mr Zuber
- O Mr Ireland
- O Mr Lam
- O Mr Lin

Student Number:__________________________

(To be used by the exam markers only.)

Question No	1-5	6	7	8	9	10	11	Total	Total %
Mark	/5	/7	/10	/4	/8	/7	/6	/47	/100

MULTIPLE CHOICE

Select the alternative A, B, C or D that best answers the question and indicate your choice by shading the appropriate letter.

1.	A	B	C	D
2.	A	B	C	D
3.	A	B	C	D
4.	A	B	C	D
5.	A	B	C	D

Objective Response

Mark your answers on the multiple choice box on the opposite page. **Marks**

1 $\dfrac{\log_{12} 8}{\log_{12} 2}$ is equal to: **1**

(A) $\log_{12} 4$ (B) $\log_{12} 6$ (C) 3 (D) $0\cdot5$

2 The derivative with respect to x of $\log\sqrt{x^2-1}$ is: **1**

(A) $\dfrac{x}{\sqrt{x^2-1}}$ (B) $\dfrac{x}{x^2-1}$ (C) $\dfrac{x}{2(x^2-1)}$ (D) $\dfrac{x}{2\sqrt{x^2-1}}$

3 The area under the curve $y=\dfrac{5}{\sqrt{x}}$, for $1\le x\le e^3$, is rotated about the x-axis.

The volume of the solid generated is: **1**

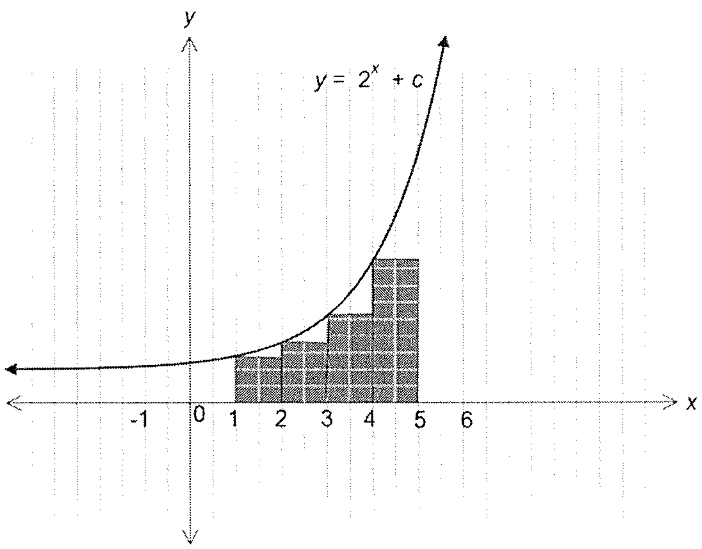
(A) 15π cubic units (B) 25π cubic units

(C) 28π cubic units (D) 75π cubic units

4 If $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx = \log_e K$, then K equals : **1**

(A) $1+e$ (B) e (C) $\dfrac{e+1}{2}$ (D) $\dfrac{(e+1)^2}{2}$

5 Consider the graph $y=2^x+c$, where c is a real number. The area of the shaded rectangles is used to find an **approximation** to the area of the region that is bounded by the graph, the x-axis and the lines $x=1$ and $x=5$.



If the total area of the shaded rectangles is 44, then the value of c is:

(A) 14 (B) -4 (C) $\dfrac{14}{5}$ (D) $\dfrac{7}{2}$ **1**

Extended Response
Write your solutions in your exam booklet, starting a new page for each Question.

Marks

Question 6 (7 marks)

(a) Find the coordinates of the point on the curve $y = 2e^{3x} + 1$ where the tangent is parallel to the line $12x - y + 1 = 0$. **3**

(b) Solve the equation $\log_3(2x-1) + \log_3(x-4) = 2$ **3**

(c) Write down the domain of the function $f(x) = \sqrt{-\log_e x}$ **1**

Question 7 (10 marks)

(a) Differentiate each of the following with respect to x:

(i) $x^3 e^{-4x}$ **2**

(ii) $\sqrt{e^x}$ **1**

(iii) $\log_e\left(\dfrac{3+2x}{5-x}\right)$ **2**

(b) Find:

(i) $\displaystyle\int \frac{3x}{4+x^2}\,dx$ **2**

(ii) $\displaystyle\int \sqrt{e^x}\,dx$ **1**

(iii) $\displaystyle\int x e^{x^2+2}\,dx$ **2**

Marks

Question 8 (4 marks)

(i) Prove using mathematical induction that

$(1\times 2)+(2\times 3)+(3\times 4)+....+n(n+1)=\frac{1}{3}n(n+1)(n+2)$, for integers $n\geq 1$. **3**

(ii) Hence evaluate $\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k(k+1)$ **1**

Question 9 (8 marks)

Given that $f(x)=\frac{e^x}{e^x+1}$,

(i) Find the first derivative of $f(x)$, and hence show that $y=f(x)$ is a monotonic increasing function for all x. **2**

(ii) What is the behaviour of $f(x)$ as $x\to\infty$ and $x\to-\infty$? **2**

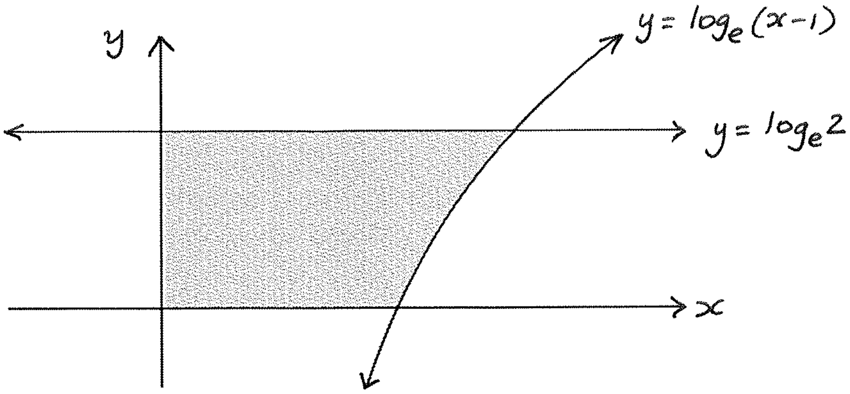
(iii) Show that the second derivative of the function is $\frac{e^x(1-e^x)}{(e^x+1)^3}$ and hence find the coordinates of any points of inflexion. **2**

(iv) Sketch the graph of $y=f(x)$, showing any intercepts, turning points, inflexions, or asymptotes. **2**

Please turn over for final questions →

Question 10 (7 marks)

(a)



Find the area enclosed by the curve $y = \log_e (x-1)$, the coordinate axes, and the line $y = \log_e 2$ (*see diagram*). **3**

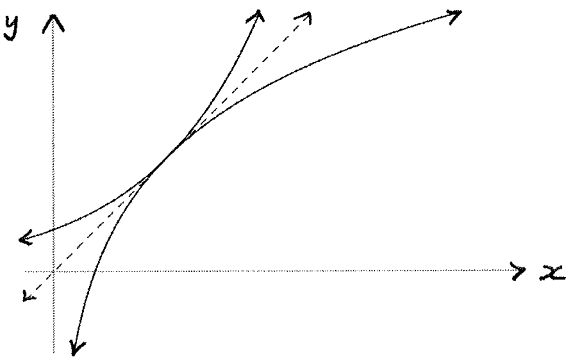
(b)

(i) Differentiate $y = x^2 \log x$ **1**

(ii) Hence evaluate $\int_1^e x \log x \, dx$ **3**

Question 11 (6 marks)

Two functions, $f(x) = a^x$ and $g(x) = \log_a x$, are drawn on the same axes so that they touch on the line $y = x$ (*see diagram*). Note that $a > 0$.



(i) Show that at the point where they touch $a^x \times \ln a = \ln x$. **1**

(ii) Write expressions for $f'(x)$ and $g'(x)$. **2**

(iii) Hence find the coordinates of the point of contact of $y = f(x)$ and $y = g(x)$. **2**

(iv) What is the value of *a*? **1**

END OF EXAM

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \; \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

1. $\dfrac{\log 8}{\log 2} = \dfrac{3\log 2}{\log 2} = 3 \quad \therefore \quad (C)$ ✓

2. $\dfrac{d}{dx} \log (x^2-1)^{\frac{1}{2}} = \dfrac{1}{\sqrt{x^2-1}} \cdot \frac{1}{2} \cdot 2x \cdot \dfrac{1}{\sqrt{x^2-1}}$

$= \dfrac{x}{x^2-1} \quad \therefore \quad (B)$ ✓

3. $V = \pi \displaystyle\int_1^{e^3} \frac{25}{x}\, dx$

$= 25\pi \left[\ln x\right]_1^{e^3} = 75\pi \quad \therefore \quad (D)$ ✓

4. $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\, dx = \left[\ln(1+e^x)\right]_0^1$

$= \ln(1+e) - \ln(1+1)$

$= \ln\left(\frac{1+e}{2}\right) \quad \therefore \quad (C)$ ✓

5. Area rectangles $= (2^1+c) + (2^2+c) + (2^3+c) + (2^4+c)$

$= 2+4+8+16 + 4c$

$= 30 + 4c$

$\therefore \ 4c = 14, \quad \therefore \ c = \frac{7}{2} \quad \therefore \quad (D)$ ✓

/5

$\boxed{Q6}$ (a) $y = 2e^{3x} + 1$

$\therefore y' = 6e^{3x}$.

Also, as $y = 12x + 1$ $\therefore$ $m = 12$

$\therefore 6e^{3x} = 12$ ✓ For this equality

$\therefore e^{3x} = 2$ $\therefore$ $x = \frac{\ln 2}{3}$

So $\therefore$ $y = 2 \cdot e^{3 \cdot \frac{\ln 2}{3}} + 1 = 2 \cdot 2 + 1$

$= 5$

$\therefore \left(\frac{\ln 2}{3}, 5\right)$. ✓✓ one each coordinate

(b) $\log_3(2x-1) + \log_3(x-4) = 2$

$\therefore \log_3\left[(2x-1)(x-4)\right] = 2$

$(2x-1)(x-4) = 9$ ✓

$2x^2 - 9x - 5 = 0$

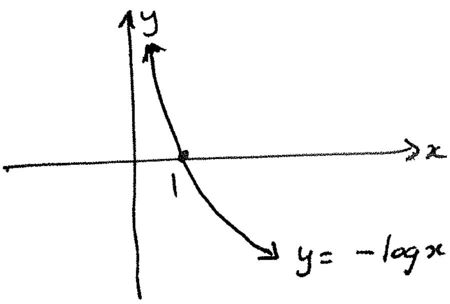
$(2x+1)(x-5) = 0$

$\therefore x = -\frac{1}{2}$ or $x = 5$ ✓

But $\log_3(-\frac{1}{2} - 4)$ not defined, ✓

$\therefore x = 5$.

(c) $f(x) = \sqrt{-\log x}$



$\therefore$ D: $0 < x \leq 1$. ✓

7

Q7

(a) (i) $y = x^3 e^{-4x}$

$\therefore y' = e^{-4x} \cdot 3x^2 + x^3 \cdot -4e^{-4x}$ ✓✓

$= x^2 e^{-4x}(3-4x)$

(ii) $y = \sqrt{e^x} = (e^x)^{\frac{1}{2}} = e^{\frac{x}{2}}$

$\left.\begin{aligned} \therefore y' &= \tfrac{1}{2} e^{\frac{x}{2}} \\ &= \tfrac{1}{2}\sqrt{e^x} \end{aligned}\right\}$ ✓

(iii) $y = \ln\left(\frac{3+2x}{5-x}\right)$

$= \ln(3+2x) - \ln(5-x)$ ✓

$\therefore y' = \frac{2}{3+2x} + \frac{1}{5-x}$ ✓

(b) (i) $\int \frac{3x}{4+x^2}\,dx = \frac{3}{2}\int \frac{2x}{4+x^2}\,dx$

$= \frac{3}{2}\ln(4+x^2) + c$ ✓✓

(ii) $\int \sqrt{e^x}\,dx = \int e^{\frac{x}{2}}\,dx$

$\left.\begin{aligned} &= 2e^{\frac{x}{2}} + c \\ &= 2\sqrt{e^x} + c \end{aligned}\right\}$ ✓

(iii) $\int x e^{x^2+2}\,dx = \frac{1}{2} e^{x^2+2} + c$ ✓✓

[delete 1 if no constant]

/10

Q8 (i) When $n=1$:

$LHS = 1(1+1) = 2$

$RHS = \cancel{\tfrac{1}{3}}(1)(1+1)(\cancel{1+2}) = 2 = LHS$

$\therefore$ true for $n=1$ ✓ must sub in

Assume true for $n=k$:

i.e. assume $1\cdot 2 + 2\cdot 3 + \dots + k(k+1) = \frac{1}{3}\cdot k(k+1)(k+2)$

Then we'd have

$1\cdot 2 + 2\cdot 3 + \dots + k(k+1) + (k+1)(k+2)$

$= \frac{1}{3}\cdot k(k+1)(k+2) + (k+1)(k+2)$ by assumption ✓

$= (k+1)(k+2)\left[\frac{1}{3}\cdot k + 1\right]$

$= (k+1)(k+2)\cdot\frac{1}{3}\left[k+3\right]$

$= \frac{1}{3}\cdot(k+1)(k+2)(k+3)$ ✓

$\therefore$ if true for $n=k$, it's true for $n=k+1$

$\therefore$ Since true for $n=1$ it is true for all $n \geq 1$ by the principle of mathematical induction.

(ii) $\lim\limits_{n\to\infty} \frac{1}{n^3}\sum\limits_{k=1}^{n} k(k+1) = \lim\limits_{n\to\infty} \dfrac{\frac{1}{3}n(n+1)(n+2)}{n^3}$ from (i)

$= \lim\limits_{n\to\infty}\left(\frac{1}{3}\cdot\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{n+2}{n}\right)$

$= \lim\limits_{n\to\infty}\left[\frac{1}{3}\cdot 1\cdot\left(1+\frac{1}{n}\right)\cdot\left(1+\frac{2}{n}\right)\right]$

$= \frac{1}{3}$ ✓

4

$\boxed{Q9}$ $f(x) = \dfrac{e^x}{e^x + 1}$

(i) $\therefore f'(x) = \dfrac{(e^x+1)\cdot e^x - e^x \cdot e^x}{(e^x+1)^2}$

$\therefore f'(x) = \dfrac{e^x}{(e^x+1)^2}$ ✓

Since $e^x > 0$ for all x, and $(e^x+1)^2 > 0$ for all x,

$\therefore f'(x) > 0$ for all x $\therefore f(x)$ is monotonic increasing. ✓

(ii) $f(x) = \dfrac{e^x}{e^x+1} = \dfrac{e^x+1-1}{e^x+1} = 1 - \dfrac{1}{e^x+1}$

$\therefore$ as $x \to \infty$, $e^x \to \infty$ $\therefore f(x) \to 1^-$ ✓

as $x \to -\infty$, $e^x \to 0$ $\therefore f(x) \to 0^+$ ✓

(iii) $f'(x) = \dfrac{e^x}{(e^x+1)^2}$

$\therefore f''(x) = \dfrac{(e^x+1)^2 \cdot e^x - e^x \cdot 2 \cdot e^x(e^x+1)}{(e^x+1)^2}$

$= \dfrac{(e^x+1)\cdot e^x - 2e^{2x}}{(e^x+1)^3}$

$= \dfrac{e^x - e^{2x}}{(e^x+1)^3}$

$\therefore f''(x) = \dfrac{e^x(1-e^x)}{(e^x+1)^3}$ ✓ (no fudging!)

→ P.T.O.

$\boxed{Q9}$ - continued

(iii) - continued:

For inflexions, $f''(x) = 0$

$\therefore\ e^x = 1 \quad \therefore \quad x = 0$

$y = \frac{1}{2}$

Test:

x	$-\ln 2$	0	$\ln 2$
$f''(x)$	$\frac{(\frac{1}{2})(1-\frac{1}{2})}{(\frac{1}{2}+1)^3}$		$\frac{2(1-2)}{(2+1)^3}$
	$+$	0	$-$

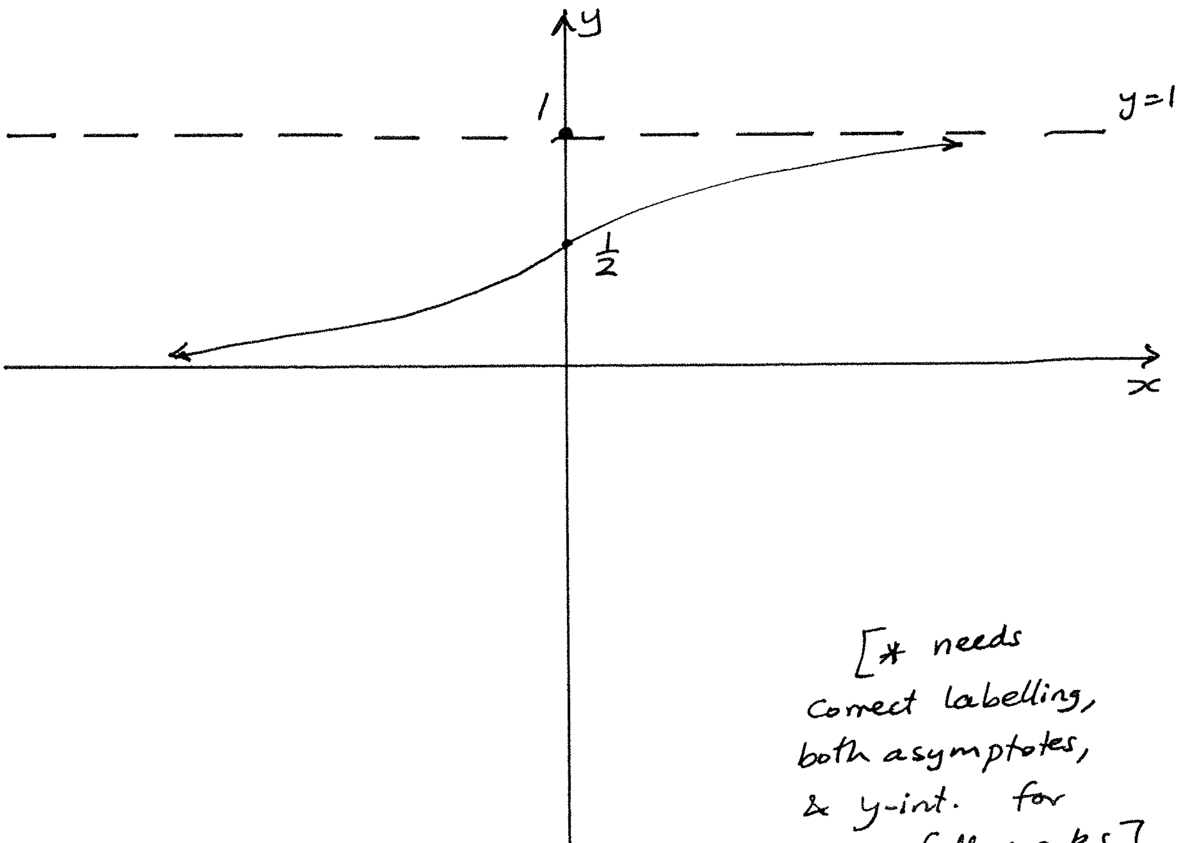
Change of concavity

$\therefore\ (0, \frac{1}{2})$ is inflexion.

needs both coords & some kind of test.

✓

(iv)



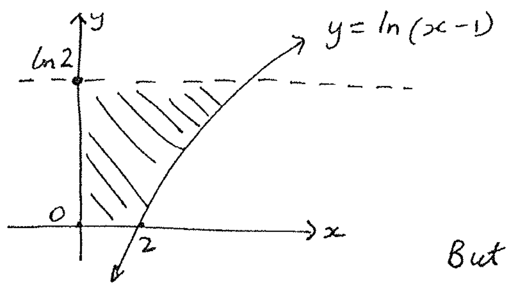
✓✓

[* needs correct labelling, both asymptotes, & y-int. for full marks]

/8

Q10

(a)



$$A = \int_0^{\ln 2} x \, dy$$

But $y = \log_e(x-1)$

$\therefore x = e^y + 1$

$$\therefore \quad A = \int_0^{\ln 2} (e^y + 1) \, dy$$

$$= \left[e^y + y\right]_0^{\ln 2}$$

$$= \left(e^{\ln 2} + \ln 2\right) - \left(e^0 + 0\right)$$

$$\therefore A = 1 + \ln 2 \text{ units}^2.$$

✓ ✓ ✓

(b) (i)
$$\left.\begin{aligned} \frac{d}{dx}\left(x^2 \log x\right) &= \log x \cdot 2x + x^2 \cdot \frac{1}{x} \\ &= 2x \log x + x \end{aligned}\right\}$$

✓

(ii) From (i),

$$\int \frac{d}{dx}(x^2 \log x)\, dx = \int 2x \log x \, dx + \int x \, dx$$

$$\left.\begin{aligned} \therefore \int_1^e x \log x \, dx &= \frac{1}{2}\left[\int_1^e \frac{d}{dx}(x^2 \log x) dx - \int_1^e x \, dx\right] \\ &= \frac{1}{2}\left[\left[x^2 \log x\right]_1^e - \left[\frac{x^2}{2}\right]_1^e\right] \end{aligned}\right\}$$

✓

$$= \frac{1}{2}\left[\left(e^2 \cdot \log e - 1^2 \cdot \log 1\right) - \left(\frac{e^2}{2} - \frac{1}{2}\right)\right]$$

✓

$$= \frac{1}{2}\left(e^2 - \frac{e^2}{2} + \frac{1}{2}\right)$$

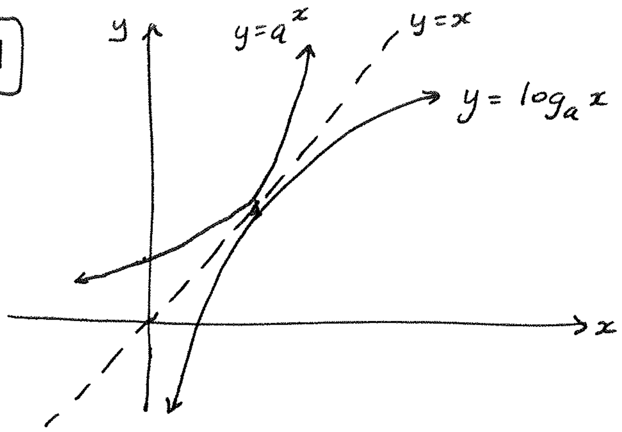
$$= \frac{e^2 + 1}{4}$$

✓

$(\doteqdot 2.097\,264)$

7

Q11



(i) when they touch, $a^x = \log_a x$

$$= \frac{\ln x}{\ln a}$$

$$\therefore \quad a^x \cdot \ln a = \ln x.$$ ✓

(ii) $f(x) = a^x = \left(e^{\ln a}\right)^x \quad \therefore \quad f'(x) = \ln a \cdot a^x$ ✓

$g(x) = \log_a x = \dfrac{\ln x}{\ln a} \quad \therefore \quad g'(x) = \dfrac{1}{\ln a \cdot x}$ ✓

(iii) They touch on $y=x$, $\therefore \; f' = g' = 1$

Since $f'(x) = 1 \quad \therefore \quad \ln a \cdot a^x = 1 \quad \therefore \; a^x = \dfrac{1}{\ln a}$

But $a^x = \dfrac{\ln x}{\ln a}$ (from (i))

$\therefore \quad \dfrac{1}{\ln a} = \dfrac{\ln x}{\ln a} \qquad \therefore \; \ln x = 1$

$x = e$
$y = e$

So point contact is (e, e). ✓✓

(iv) Since $g'(e) = 1 \quad \therefore \quad \dfrac{1}{\ln a \cdot e} = 1$

$\therefore \; \ln a = \dfrac{1}{e} \qquad \therefore \quad a = e^{\frac{1}{e}}.$ ✓

(other routes/working possible).

/6