North Sydney Boys High School

EXTENSION 1 MATHEMATICS

2016 HSC Course Assessment Task 2

16th March 2016

General Instructions

- Working time – 1 hours
  (plus 5 minutes reading time).
- Use a blue or black pen.
  Diagrams may be sketched in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt all questions.

Section I

- Mark answers on the answer grid provided.

Section II

- Commence each question on a new page.
- Write on both sides of the paper.
- Show all necessary working for every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED:** ....

Teacher (please ✓)

☐ Mr Lin ☐ Mr Jooma ☐ Mr Ireland

☐ Mr Berry ☐ Ms Ziaziaris ☐ Ms Lee

Question	1-5	6	7	8	9	10	11	Total	%
Marks	$\overline{5}$	$\overline{7}$	$\overline{7}$	$\overline{5}$	$\overline{7}$	$\overline{7}$	$\overline{8}$	$\overline{46}$	$\overline{100}$

Section I

5 Marks

Attempt Questions 1 to 5

Mark your answers on the answer sheet provided on page 2.

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1. Find the value of $\log_4 32$

A) 1.5

B) 2

C) 2.5

D) 3

2. The solution to $\log_3(x-1) = 3$ is?

A) 7

B) 10

C) 26

D) 28

3. Evaluate $\int_e^{e^2} \frac{2}{x} dx$

A) 2

B) 4

C) 6

D) 8

4. The derivative of $y = x \tan 2x$ is

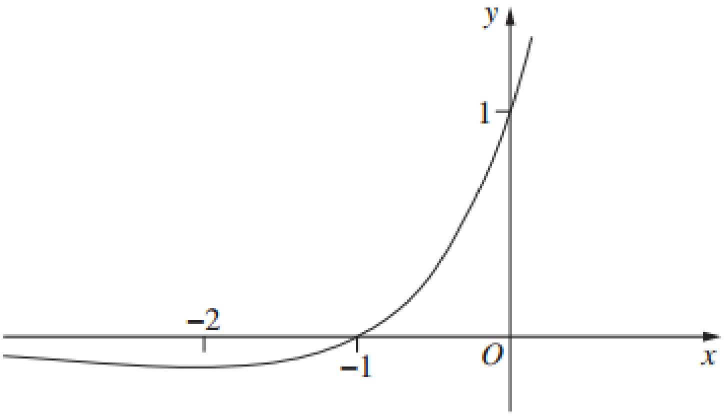
(A) $\tan 2x + x \sec^2 2x$

(B) $\tan 2x + 2 \sec^2 2x$

(C) $x \tan 2x + \sec^2 2x$

(D) $2 \tan 2x + x \sec^2 2x$

5. The diagram shows the graph of $y = e^x(1 + x)$



How many solutions are there to the equation $e^x(1 + x) = 1 - x^2$?

(A) 0

(B) 1

(C) 2

(D) 3

END OF SECTION I

1	A	B	C	D
2	A	B	C	D
3	A	B	C	D
4	A	B	C	D
5	A	B	C	D

Section II

Total of **41** marks

Attempt Questions 6 to 11.

Write your answers in the writing book provided.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (7 marks) Use a **NEW** page.

Differentiate the following

(a) $y = e^{\sin x}$ **1**

(b) $y = \cos(\log x)$ **2**

(c) $y = e^{x^2} e^{x}$ **2**

(d) $y = \log\left(\dfrac{x}{x^2 - 1}\right)$ **2**

Question 7 (7 marks) Use a **NEW** page.

(a) Find the following integrals

(i) $\displaystyle\int \frac{2x + 3}{x^2 + 3x + 5}\, dx$ **1**

(ii) $\displaystyle\int \frac{x^2 + 2x - 3}{x^2}\, dx$ **2**

(b) Evaluate the following integrals

(i) $\displaystyle\int_0^1 x e^{x^2 + 3}\, dx$ **2**

(ii) $\displaystyle\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$ **2**

Question 8 (5 marks) Use a **NEW** page.

(a) The area of a sector of a circle of radius 6cm is 50cm^2. Find the length of the arc. **2**

(b) Find the equation of the tangent to the curve $y = 2\sin(2x + \frac{\pi}{3})$ at the point where $x = \frac{\pi}{2}$. **3**

Question 9 (7 marks) Use a **NEW** page.

(a) Find the volume of the solid generated when the curve $y = \tan x$, the x axis and $x = \frac{\pi}{3}$ is rotated about the x axis. **3**

(b) Let $f(x) = 2\cos 2x$

(i) Sketch $f(x)$ for $0 \le x \le 2\pi$ **2**

(ii) Hence, find all values of x such that $f(x) \le 1$, for $0 \le x \le 2\pi$ **2**

Question 10 (7 marks) Use a **NEW** page.

(a) (i) Show that $\frac{1}{2^2-1} + \frac{1}{3^2-1} + \cdots + \frac{1}{p^2-1} = \frac{p(3p+5)}{4p(p+1)}$ by using the principle of mathematical induction **3**

(ii) Hence find **1**

$$\lim_{x\to\infty} \sum_{t=2}^{x} \left(\frac{1}{t^2 - 1}\right)$$

(b) If $(ax)^{\log a} = (bx)^{\log b}$, prove that $x = \frac{1}{ab}$ **3**

Question 11 (8 marks) Use a **NEW** Page.

Given the function $y = x^x$ for $x > 0$

(a) What is the value of $\lim_{x\to 0} x^x$? **1**

(b) By using $x = e^{\log x}$ or otherwise, show that $\frac{dy}{dx} = x^x(1 + \log x)$ **2**

(c) Find the turning point(s) and determine its nature **3**

(d) Using the information from above, sketch the curve of $y = x^x$ **2**

End of Examination

Multiple Choice

1. C 2. D 3. A 4. B 5. C

Question 6.

a) $y = e^{\sin x}$

$\frac{dy}{dx} = \cos x \cdot e^{\sin x}$

b) $y = \cos(\log x)$

$\frac{dy}{dx} = \frac{1}{x} \cdot -\sin(\log x)$

$= -\frac{\sin(\log x)}{x}$

(c) $y = e^{x^2} \cdot e^{x}$

$= e^{x^2+x}$

$\frac{dy}{dx} = (2x+1)e^{x^2+x}$

(d) $y = \log \frac{x}{x^2-1}$

$= \log x - \log(x^2-1)$

$\frac{dy}{dx} = \frac{1}{x} - \frac{2x}{x^2-1}$

Question 7

a) (i) $\ln(x^2+3x+5)+C$

(ii) $\int \frac{x^2+2x-3}{x^2}\,dx$

$= \int 1 - \frac{2}{x} - \frac{3}{x^2}\,dx$

$= x - 2\ln|x| + \frac{3}{x} + C$

b) (i) $\int_0^1 x e^{x^2+3}\,dx$

$= \frac{1}{2}\left[e^{x^2+3}\right]_0^1$

$= \frac{1}{2}e^3[e-1]$

(ii) $\int_0^1 \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

$= \left[\ln(e^x+e^{-x})\right]_0^1$

$= \ln\left(e+\frac{1}{e}\right) - \ln 2$

$= \ln\left(\frac{e^2+1}{2e}\right)$

Question 8

a) $A = \frac{1}{2} r^2 \theta$

$50 = \frac{1}{2} \cdot 36 \cdot \theta$

$\therefore \theta = \frac{25}{9}$

$\therefore l = r\theta$

$= 6 \times \frac{25}{9}$

$= \frac{150}{9}$ cm

$= \frac{50}{3}$ cm

b) $y = \sin\left(x + \frac{\pi}{3}\right)$

$\frac{dy}{dx} = \cos\left(x + \frac{\pi}{3}\right)$

when $x = \frac{\pi}{2}$ $y = \frac{1}{2}$

$m = \cos\left(\frac{5\pi}{6}\right)$

$= -\frac{\sqrt{3}}{2}$

$\therefore y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{2}\right)$

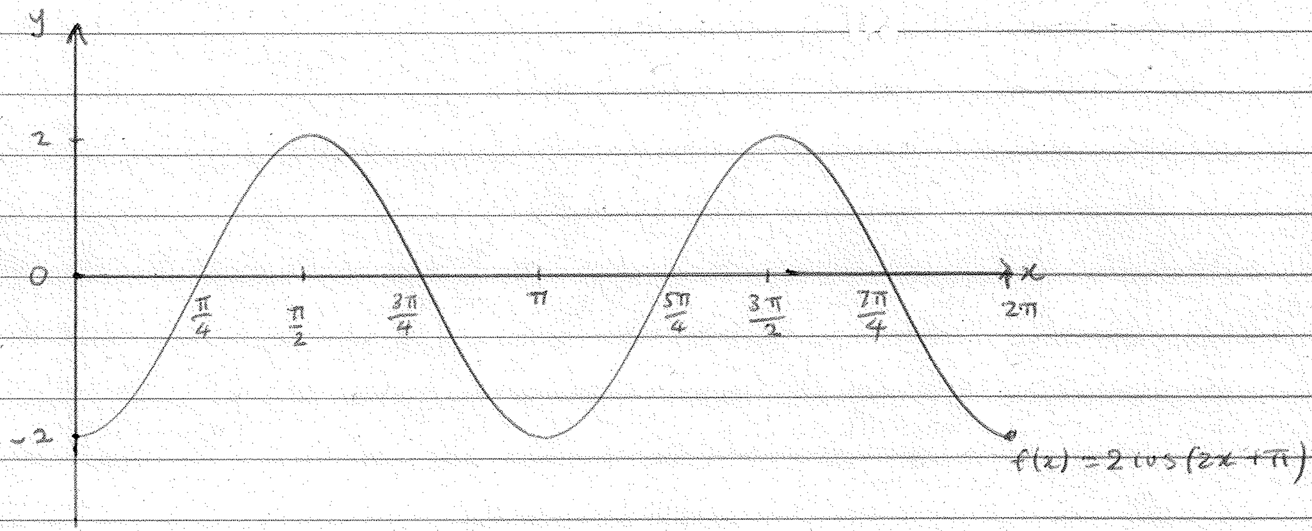
$-2y + 1 = \sqrt{3}x + \frac{\sqrt{3}}{2}$

$2\sqrt{3}x + 4y + (\sqrt{3} - 2) = 0$

Question 9

a) $f(x) = 2\cos(2x + \pi)$

$= -2\cos(2x)$



b) $(ax)^{\log a} = (bx)^{\log b}$

$\log (ax)^{\log a} = \log (bx)^{\log b}$

$\log a \cdot \log(ax) = \log b \cdot \log(bx)$

$\log a\,(\log a + \log x) = \log b\,(\log b + \log x)$

$(\log a)^2 + \log a \cdot \log x = (\log b)^2 + \log b \cdot \log x$

$(\log a)^2 - (\log b)^2 = (\log b - \log a) \cdot \log x$

$(\log a - \log b)(\log a + \log b) = \log x \cdot (\log b - \log a)$

$\log(ab) = -\log x$

$\log x = \log\left(\frac{1}{ab}\right)$

$x = \frac{1}{ab}$

Question 10

Prove true for the base case: $p=2$.

LHS $= \frac{1}{3}$ RHS $= \frac{(2-1)(6+2)}{8(2+1)}$

$= \frac{1}{3}$

$\therefore$ true for $p=2$

Assume true for $p=k$ where $k \geq 2$, $k \in \mathbb{Z}$.

i.e. $\frac{1}{2^2-1} + \frac{1}{3^2-1} + \dots + \frac{1}{k^2-1} = \frac{(k-1)(3k+2)}{4k(k+1)}$ — (*)

Required to prove true for $p=k+1$

i.e. $\frac{1}{2^2-1} + \frac{1}{3^2-1} + \dots + \frac{1}{k^2-1} + \frac{1}{(k+1)^2-1} = \frac{k(3k+5)}{4(k+1)(k+2)}$

Now LHS $= \frac{(k-1)(3k+2)}{4k(k+1)} + \frac{1}{(k+1)^2-1}$ from the induction hypothesis (*)

$= \frac{(k-1)(3k+2)}{4k(k+1)} + \frac{1}{k(k+2)}$

$= \frac{(k-1)(3k+2)(k+2) + 4(k+1)}{4k(k+1)(k+2)}$

$= \frac{3k^3 + 5k^2}{4k(k+1)(k+2)}$

$= \frac{k(3k+5)}{4(k+1)(k+2)}$

$=$ RHS

$\therefore$ true for $p=k+1$

$\therefore$ Since its true for $p=2$ and $p=k+1$, then it is true for all integers $p \geq 2$ by the principle of mathematical induction.

Question 11

a) $\lim_{x \to 0} x^x = 1$

b) $y = x^x = e^{x \log_e x}$

$$\frac{dy}{dx} = \frac{d}{dx}(x \log_e x) \cdot e^{x \log_e x}$$

$$= \left(x \cdot \frac{1}{x} + \log_e x\right) \cdot e^{x \log_e x}$$

$$= (1 + \log_e x) \cdot x^x$$

c) stationary point occurs when $\frac{dy}{dx} = 0 \Rightarrow (1 + \log x)\, x^x = 0$

since $x^x \neq 0$ for all values.

$1 + \log x = 0$

$x = \frac{1}{e} \approx 0.368$

$y = \left(\frac{1}{e}\right)^{\frac{1}{e}} \approx 0.692$

x	0.3	$\frac{1}{e}$	0.4
$\frac{dy}{dx}$	-0.14	0	0.06

$\therefore$ minimum turning point at $\left(\frac{1}{e}, \left(\frac{1}{e}\right)^{\frac{1}{e}}\right)$

d)