NORTH SYDNEY GIRLS HIGH SCHOOL

YEAR 12 – TERM 1 ASSESSMENT

2005

MATHEMATICS

EXTENSION COURSE 1

TIME ALLOWED: 60 minutes
Plus 2 minutes reading time

INSTRUCTIONS:

* Start each question on a new page
* Hand each question in separately, including a sheet for non-attempts.
* Show all necessary working.

This task is worth 20% of the HSC Assessment Mark

Question One – (8 marks)

Marks

a) Simplify $\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6}$. 2

b) Using the expansion of tan (A + B) express tan 75° in its simplest exact form 3

c) Evaluate the definite integral $\int_0^{\frac{\pi}{4}} \cos^2 x\, dx$ 3

Question Two – (9 marks)

a) If $\tan\frac{\theta}{2} = t$, express $1 + \tan\theta \tan\frac{\theta}{2}$ *in terms of* t 2

b) Prove the following identity $\frac{\sin 2\theta - \sin\theta}{\cos 2\theta - \cos\theta + 1} \equiv \tan\theta$ 3

c) The line $y = mx$ makes an angle of 45° with the line $y = 3x$. 4

Show that $|m-3| = |1+3m|$. Hence find the equations of the lines $y = mx$ which make an angle of 45° with the line $y = 3x$.

Question Three – (8 marks)

a) Water is flowing out of a tank at the rate $\frac{dV}{dt} = 10t - 250$, where V is the volume in litres remaining in the tank at time t minutes after time zero.

(i) when does the water stop flowing? 1

(ii) Given that the tank has 20 litres of water left in it when the water flow stops, find an equation for the volume at any time t. 2

Marks

b) The position of a particle P, moving in a straight line at any time t is given by $x = 2\sin t - t, t \geq 0$.

(i) Find an expression for the velocity of the particle at any time t. 1

(ii) Determine the first time the particle comes to rest. 1

(iii) Calculate the total distance travelled by the particle in the first π seconds. 3

Question Four (8 marks)

a) Find all angles for which $5 \sin \theta^\circ - 12 \cos \theta^\circ = 12 \quad 0^\circ \leq \theta \leq 360^\circ$ 4

b) A sphere is expanding so that its surface area is increasing at the rate of $4cm^2/min$. Find the rate of change of the volume of the sphere at the instant when the radius of the sphere is 20cm. 4

Question Five (9 marks)

a) L is the fixed point (-4, 7). P $(2ap, ap^2)$ is a variable point on the parabola $x^2 = 4ay$. M is the midpoint of LP. Find the equation of the locus of M as P varies on the parabola. 4

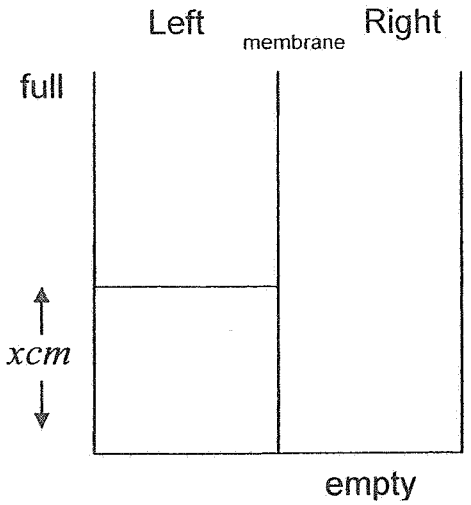
b) (i) Write down the equation of the chord of contact of the tangents from the external point T (x_1, y_1) to the parabola $x^2 = 12y$. 1

(ii) This chord of contact, when produced meets the directrix of the parabola at R, prove that RT subtends a right angle at the focus. 4

Marks

Question Six (7 marks)

A Chamber is divided into 2 identical parts by a porous membrane. The left compartment is initially full and the right compartment is empty. The liquid is let through at a rate proportional to the difference between the level in the left compartment, $x$cm and the average level such that $\frac{dx}{dt} = k(25 - x)$



(i) Show that $x = 25 + Ae^{-kt}$ is a solution to this equation. **2**

(ii) What value does the level in the left compartment approach? **1**

(iii) Explain briefly why the initial height is 50cm. **1**

(iv) Find the value of A **1**

(v) the liquid in the left compartment has fallen by 10cm in 5 minutes, find the value of k (leave your answer in exact form). **2**

END OF TEST

Q5 (b) (i) Show that the derivative of $y = \log_e(\cos x)$ is $-\tan x$. 6

(ii) Using the same set of axes sketch the curves $y = \tan x$ and $y = \sin 2x$ in the domain $O \le x \le \frac{\pi}{2}$

(iii) Show that the curves in part (ii) intersect at $x = \frac{\pi}{4}$.

(iv) Calculate the area enclosed between the two curves in the domain $O \le x \le \frac{\pi}{2}$. *(Leave your answer in exact form).*

Q6 (b) (i) Write down the period and amplitude for $y = 3\cos\frac{x}{2}$. 3

(ii) Sketch the curve $y = 3\cos\frac{x}{2}$ for $-2\pi \le x \le 2\pi$

(c) The daily growth of the population of a colony of insects is 10% of the excess of the population over 1.2×10^6, 5

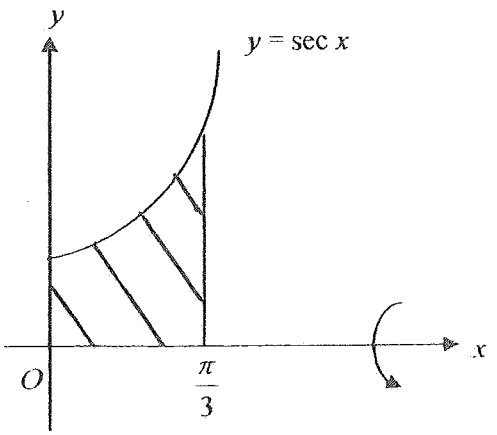
ie $\frac{dP}{dt} = 0.1(P - 1.2 \times 10^6)$. At $t = 0$, the population is 2.7×10^6

(i) Show that $P = 1.2 \times 10^6 + Ae^{0.1t}$ satisfies $\frac{dP}{dt} = 0.1(P - 1.2 \times 10^6)$.

(ii) Determine the population after $3\frac{1}{2}$ days

(iii) If a scientist checks the population each day, which is the first day on which she should notice the original population has tripled?

Q5 b)



In the diagram, the shaded region is bounded by the curve $y = \sec x$, 3
the coordinate axes and the line $x = \frac{\pi}{3}$.

The shaded region is rotated about the x-axis.
Calculate the volume of the solid of revolution formed.

SOLUTIONS

YEAR 12 T1 EXTENSION 1 MATHEMATICS.

Question one

a) $\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6}$

$= \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$

$= \cos\left(\frac{\pi}{2}\right)$

$= 0$

b) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(30°+45°)$

$= \frac{\tan 30° + \tan 45°}{1 - \tan 30° \tan 45°}$

$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \left(\frac{1}{\sqrt{3}}\right)(1)}$

$= \frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$

$= \frac{1+\sqrt{3}}{\sqrt{3}-1}$

c) $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$

$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(\cos 2x + 1) \, dx$

$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left(\left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4}\right) - \left(\frac{1}{2}\sin 0 - 0\right)\right)$

$= \frac{1}{2}\left(\left(\frac{1}{2} + \frac{\pi}{4}\right) - (0)\right)$

$= \frac{1}{4} + \frac{\pi}{8}$

Question two

a) $\tan\frac{\theta}{2} = t$.

$1 + \tan\theta\tan\frac{\theta}{2}$

$= 1 + \left(\frac{2t}{1-t^2}\right)(t)$

$= 1 + \frac{2t^2}{1-t^2}$.

$= \frac{1-t^2+2t^2}{1-t^2}$

$= \frac{1+t^2}{1-t^2}$

b) LHS $= \frac{\sin 2\theta - \sin\theta}{\cos 2\theta - \cos\theta + 1}$

$= \frac{2\sin\theta\cos\theta - \sin\theta}{2\cos^2\theta - 1 - \cos\theta + 1}$

$= \frac{2\sin\theta\cos\theta - \sin\theta}{2\cos^2\theta - \cos\theta}$

$= \frac{\sin\theta(2\cos\theta - 1)}{\cos\theta(2\cos\theta - 1)}$

$= \frac{\sin\theta}{\cos\theta}$.

$= \tan\theta$

$=$ RHS.

Question two Continued

) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

line 1: $y = mx$, $m_1 = m$

line 2: $y = 3x$, $m = 3$

$$\tan 45^\circ = \left|\frac{m-3}{1+3m}\right|$$

$$1 = \left|\frac{m-3}{1+3m}\right|$$

$$|1+3m| = |m-3|$$

$$|m-3| = |1+3m|$$

$m - 3 = 1 + 3m$

$-2m = 4$

$m = -2$

or

$m - 3 = -(1+3m)$

$m - 3 = -1 - 3m$

$4m = 2$

$m = \frac{1}{2}$

∴ the lines are

$y = -2x$

$y = \frac{1}{2}x$

Question three.

a) $\frac{dV}{dt} = 10t - 250$

$\frac{dV}{dt} = 0$

$0 = 10t - 250$

$10t = 250$

$t = 25$

flow stops after 25 minutes

(ii) $t = 25$, $V = 20$.

$\frac{dV}{dt} = 10t - 25$

$V = 5t^2 - 250t + C$

$20 = 5(25)^2 - 250(25) + C$

$C = 3145$

$V = 5t^2 - 250t + 3145.$

b) $x = 2\sin t - t$, $t \geq 0$

(i) $v = \frac{dx}{dt} = 2\cos t - 1$

(ii) $0 = 2\cos t - 1$

$2\cos t = 1$

$\cos t = \frac{1}{2}$

$t = \frac{\pi}{3}$

particle at rest at time equals $\frac{\pi}{3}$ seconds

Question three continued

b iii)

$t = 0$, $x = 2\sin 0 - 0$

$= 0$

$t = \frac{\pi}{3}$, $x = 2\sin\frac{\pi}{3} - \frac{\pi}{3}$

$= 2 \times \frac{\sqrt{3}}{2} - \frac{\pi}{3}$

$= \sqrt{3} - \frac{\pi}{3}$

$t = \pi$, $x = 2\sin\pi - \pi$

$= -\pi$.

total distance travelled

$= \left|\left(\sqrt{3} - \frac{\pi}{3}\right) - 0\right| + \left|-\pi - \left(\sqrt{3} - \frac{\pi}{3}\right)\right|$

$= \left|\sqrt{3} - \frac{\pi}{3}\right| + \left|-\frac{2\pi}{3} - \sqrt{3}\right|$

$= 2\sqrt{3} + \frac{\pi}{3}$

Question four

a) $5\sin\theta^\circ - 12\cos\theta^\circ = 12$

$5\left(\frac{2t}{1+t^2}\right) - 12\left(\frac{1-t^2}{1+t^2}\right) = 12.$

$5(2t) - 12(1-t^2) = 12(1+t^2)$

$10t - 12 + 12t^2 = 12 + 12t^2$

$10t = 24$

$t = \frac{12}{5}$

$\tan\frac{\theta}{2} = \frac{12}{5}$

$\frac{\theta}{2} = 67.380^\circ$

$\theta = 134^\circ 46'$

check 180°

$5\sin 180^\circ - 12\cos 180^\circ = 12$

$5(0) - 12(-1) = 12$ ✓

$\therefore \theta = 134^\circ 46', 180^\circ$

Question four.

b) $\frac{dS}{dt} = 4\text{cm}^2/\text{min}$

$V = \frac{4}{3}\pi r^3$ $\quad S = 4\pi r^2$

$\frac{dV}{dr} = 4\pi r^2$ $\quad \frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}$

$4 = 8\pi r \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{2\pi r}$

$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}.$

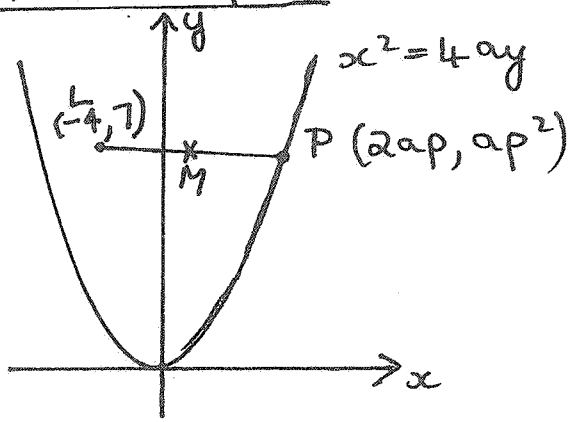
$\frac{dV}{dt} = 4\pi r^2 \times \frac{1}{2\pi r}$

$= 2r$

$r = 20$, $\frac{dV}{dt} = 2(20)$

rate of change of Volume is $40\ \text{cm}^3/\text{min}$.

Question five



Question five continued.

Coordinates of M

$$x = \frac{x_1 + x_2}{2}$$

$$= -\frac{4 + 2ap}{2}$$

$$x = ap - 2 \quad (1)$$

$$y = \frac{y_1 + y_2}{2}$$

$$y = \frac{7 + ap^2}{2} \quad (2)$$

from (1)

$$x = ap - 2$$

$$ap = x + 2$$

$$p = \frac{x+2}{a}$$

Subsitute in (2)

$$y = \frac{7 + a\left(\frac{x+2}{a}\right)^2}{2}$$

$$2y = 7 + a \times \frac{(x+2)^2}{a^2}$$

$$2y = 7 + \frac{(x+2)^2}{a}$$

$$2ay = 7a + (x+2)^2$$

$$(x+2)^2 = 2ay - 7a$$

Question five continued

b)(i) chord of contact

$$xx_1 = 2a(y + y_1)$$

$$xx_1 = 6(y + y_1) \quad (1)$$

(ii) Equation of the directrix

$$y = -3 \quad (2)$$

Sub (2) into (1)

$$xx_1 = 6(-3 + y_1)$$

$$x = \frac{6(-3 + y_1)}{x_1}$$

$$= \frac{-18 + 6y_1}{x_1}$$

$\therefore R\left(\frac{-18+6y_1}{x_1}, -3\right)$

focus $(0, 3)$ S

gradient RS (m_1)

$$m_1 = \frac{3 - -3}{0 - \frac{-18+6y_1}{x_1}}$$

$$= \frac{6}{\frac{18 - 6y_1}{x_1}}$$

$$= \frac{6x_1}{18 - 6y_1}$$

$$= \frac{x_1}{3 - y_1}$$

gradient ST (m_2)

$$m_2 = \frac{y_1 - 3}{x_1 - 0}$$

$$= \frac{y_1 - 3}{x_1}$$

for right angle

$$m_1 \cdot m_2 = -1$$

$$\frac{x_1}{3 - y_1} \times \frac{y_1 - 3}{x_1}$$

$$= \frac{x_1}{3 - y_1} \times -\frac{(3 - y_1)}{x_1}$$

$$= -1$$

$\therefore \angle RST$ is 90°

Question six

(i) $x = 25 + Ae^{-kt} \quad \Rightarrow Ae^{-kt} = x - 25$

$\frac{dx}{dt} = -kAe^{-kt}$

$\frac{dx}{dt} = -k(x-25)$

$\frac{dx}{dt} = k(25-x)$

(ii) $t \to \infty, \quad x = 25 + Ae^{-kt}$

$Ae^{-kt} \to 0$

$\therefore x \to 25$

(iii) $t \to \infty, \quad x \to$ Average

Left = full Right = 0

$\frac{\text{full} + 0}{2} = 25$

full = 50

$\therefore$ initial height is 50 cm

(iv) $x = 25 + Ae^{-kt}$

$t = 0, \quad x = 50$

$50 = 25 + Ae^{0}$

$A = 25$

$\therefore x = 25 + 25e^{-kt}$

(v) $t = 5, \quad x = 40$ cm

$40 = 25 + 25e^{-k(5)}$

$15 = 25e^{-5k}$

$\frac{3}{5} = e^{-5k}$

$\log_e \frac{3}{5} = -5k$

$k = -\frac{1}{5}\log_e \frac{3}{5}$

Markers please record marks as

TRIG - Q 1, 2, 4 a /21

CALCULUS APP - Q 3, 4b, 6 /19

PARABOLA - Q 5 /9

* Question four a)

$5\sin\theta - 12\cos\theta$

$\equiv A\sin(\theta - \alpha)$

$= A(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

$= A\sin\theta\cos\alpha - A\cos\theta\sin\alpha$

$A\cos\alpha = 5 \quad (1)$

$A\sin\alpha = 12 \quad (2)$

$\frac{(2)}{(1)} \quad \frac{\sin\alpha}{\cos\alpha} = \frac{12}{5}$

$\tan\alpha = \frac{12}{5}$

$\alpha \doteqdot 67{\cdot}380^\circ$

$(1)^2 + (2)^2$

$A^2(\cos^2\alpha + \sin^2\alpha) = 5^2 + 12^2$

$A^2 = 13^2$

$A = \pm 13, \quad A > 0$

$5\sin\theta - 12\cos\theta$

$= 13(\sin(\theta - 67{\cdot}380))$

$12 = 13\sin(\theta - 67{\cdot}380)$

$\sin(\theta - 67{\cdot}380) = \frac{12}{13}$

$\theta - 67{\cdot}380 = 67{\cdot}380 \text{ or } 112{\cdot}62$

$\theta = 134^\circ 46' \text{ or } 180^\circ.$

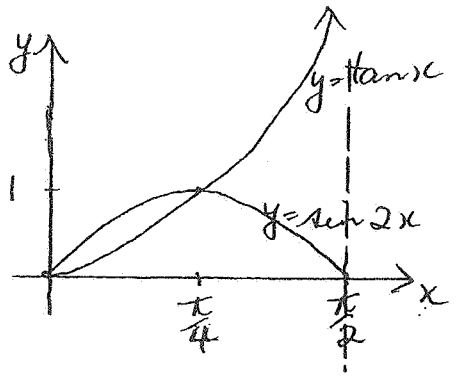
EXTRA QUESTIONS.

Q5 Y12 T1 2004

(b)(i) $\frac{dy}{dx} = \frac{-\sin x}{\cos x}$ 1mk

$= -\tan x$

(ii)



1mk

(iii) $\tan\frac{\pi}{4} = 1$

$\sin 2\times\frac{\pi}{4} = 1$ 1mk

(iv) $A = \int_0^{\pi/4} (\sin 2x - \tan x)\,dx$ 1mk

$= \left[-\frac{\cos 2x}{2} + \log_e(\cos x)\right]_0^{\pi/4}$ 1mk

$= 0 + \log_e\sqrt{2} + \frac{1}{2}$

$= \frac{1}{2}(\log_e 2 + 1)$ 1mk

Q6

(c)(i) $P = 1.2\times10^6 + Ae^{0.1t}$

$Ae^{0.1t} = P - 1.2\times10^6$

$\frac{dP}{dt} = 0.1Ae^{0.1t}$ 1mk

$= 0.1(P - 1.2\times10^6)$

(ii) when $t=0$, $P = 2.7\times10^6$

$2.7\times10^6 = 1.2\times10^6 + Ae^0$

$A = 1500000$ 1mk

when $t = 3.5$

$P = 1.2\times10^6 + 1.5\times10^6 e^{0.35}$

$= 3328601.323$ 1mk

(iii) $8.1\times10^6 = 1.2\times10^6 + 1.5\times10^6 e^{0.1t}$ ↑1mk

$4.6 = e^{0.1t}$

$t = \frac{\ln 4.6}{0.1}$

$\doteqdot 15.26$ 1mk

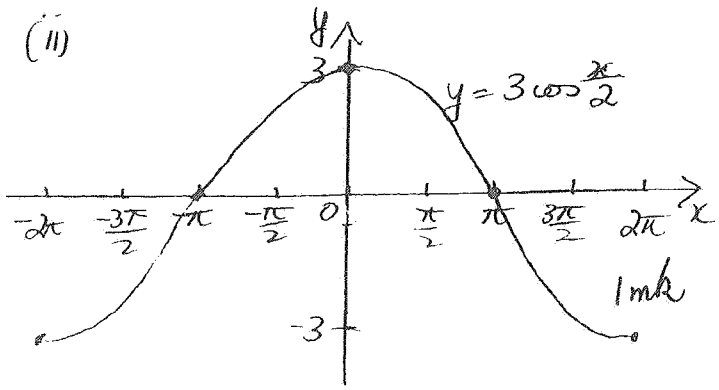
15th or 16th day

Q6

(b)(i) Period 4π 1mk

Amplitude 3 1mk

(ii)



1mk

Q5 Y11 T4 2004

b)

$V = \pi\int_0^{\pi/3} \sec^2 x\,dx$

$(3m) = \pi\left[\tan x\right]_0^{\pi/3}$

$= \pi\left[\tan\frac{\pi}{3} - \tan 0\right]$

$= \pi\cdot\sqrt{3}$

$= \pi\sqrt{3}$ cubic units