NORTH SYDNEY GIRLS HIGH SCHOOL

YEAR 12 – TERM 1 ASSESSMENT

2006

MATHEMATICS EXTENSION 1

TIME ALLOWED: 60 minutes
Plus 2 minutes reading time

INSTRUCTIONS:

- Start each question on a new page
- Hand each question in separately, including a sheet for non-attempts
- Show all necessary working

This task is worth 20% of the HSC Assessment Mark

Question One (9 Marks) Marks

(a) What is the exact value of $\cos\left(\frac{\pi}{6}\right)$? **1**

(b) Differentiate $\cos(x^2+1)$ **1**

(c) Find $\int \sec^2 5x \, dx$ **1**

(d) (i) Sketch the curve $y = 4\sin 2x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ **2**

(ii) On your diagram for part (i), sketch the line $y = \frac{1}{3}x$, and shade the region represented by $\int_0^{\frac{\pi}{4}} \left(4\sin 2x - \frac{1}{3}x\right) dx$ **2**

(iii) Find the exact value of the integral in part (ii). **2**

Question Two (8 Marks)

(a) A sphere is being heated so that its surface area is increasing at a constant rate of 25 cm^2 per second. Find the rate of increase of the volume when the radius is 5 cm. **3**

(b) One hundred grams of cane sugar in water are being converted into dextrose at a rate which is proportional to the amount unconverted at any time,

i.e. if M grams are converted in t minutes, then $\frac{dM}{dt} = k\,(100 - M)$,

where k is a constant.

(i) Show that $M = 100 + Ae^{-kt}$, where A is a constant which satisfies the above equation. **2**

(ii) Find the value of A (initially no cane sugar has been converted to dextrose) **1**

(iii) If 40 grams are converted in the first 15 minutes, find how many grams are converted in the first 30 minutes. **2**

Question Three (10 Marks)

(a) Evaluate $\lim_{x \to 0} \frac{\sin\left(\frac{x}{4}\right)}{3x}$ **2**

(b) (i) Express $\cos x - \sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$ **2**

(ii) Hence, or otherwise, solve the equation $\cos x - \sin x = \frac{\sqrt{2}}{2}$ for $0 \le x \le 2\pi$ **2**

(c) Prove $\frac{\tan A}{\tan 2A - \tan A} \equiv \cos 2A$ **4**

Question Four (10 Marks)

(a) Find the exact value of $\sin 105°$ **2**

(b) Find the volume of the solid generated when $y = \sin 3x$ is rotated around the x axis from $x = 0$ to $x = \frac{\pi}{3}$. **4**

(c) Differentiate $x \sin 3x$ with respect to x and hence evaluate $\int_0^{\frac{\pi}{2}} x\cos 3x\,dx$ **4**

Question Five (9 Marks)

(a) If $y = \tan 2x$, find the equation of the tangent to the curve at $x = \frac{\pi}{6}$ **3**

(b) Find the acute angle between the lines $4x + y + 5 = 0$ and $6x + 3y - 7 = 0$ correct to the nearest minute. **3**

(c) Solve the equation $\sin 2\theta + \cos\theta = 0$ for $0 \le \theta \le 2\pi$ **3**

Question Six (10 Marks)

(a) Solve $5\sin\theta - 2\cos\theta = 2$ for $0° \le \theta \le 360°$, using the result that $\tan\frac{\theta}{2} = t$ **4**

(b) A particle moves along a straight line so that its displacement, x metres, from a fixed point O is given by $x = 1 - 3\cos\left(\frac{t}{2}\right)$, where t is measured in seconds.

(i) Sketch the graph of x as a function of t for $0 \le t \le 4\pi$ **2**

(ii) Hence, or otherwise, find when and where the particle first comes to rest after $t = 0$ **2**

(iii) Find a time when the particle reaches its maximum speed. What is this speed? **2**

End of paper

Solutions

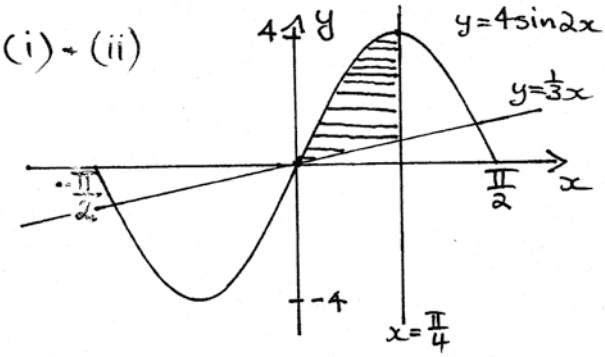
Question one

(a) $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(b) $\frac{d}{dx}\left(\cos(x^2+1)\right)$
$= -2x\sin(x^2+1)$

(c) $\int \sec^2 5x\,dx = \frac{1}{5}\tan 5x + C$

(d) (i) + (ii)



(iii) $\int_0^{\frac{\pi}{4}} 4\sin 2x - \frac{1}{3}x\,dx$

$= \left[-2\cos 2x - \frac{1}{3}\cdot\frac{x^2}{2}\right]_0^{\frac{\pi}{4}}$

$= \left(-2\cos 2\left(\frac{\pi}{4}\right) - \frac{1}{3}\cdot\frac{\left(\frac{\pi}{4}\right)^2}{2}\right) - \left(-2\cos 2(0) - \frac{1}{3}\cdot\frac{(0)^2}{2}\right)$

$= \left(-2\cos\frac{\pi}{2} - \frac{\frac{\pi^2}{16}}{6}\right) - (-2-0)$

$= \left(0 - \frac{\pi^2}{96}\right) - (-2)$

$= 2 - \frac{\pi^2}{96}$

Question two

(a) $\frac{dA}{dt} = 25\,cm^2/s$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt}$

$25 = 8\pi r\cdot\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{25}{8\pi r}$

$r=5,\ \frac{dr}{dt} = \frac{25}{8\pi\times 5}$
$= \frac{5}{8\pi}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr}\,\frac{dr}{dt}\cdot$

$= 4\pi r^2\,\frac{dr}{dt}\cdot$

$r=5,\ \frac{dV}{dt} = 4\pi(5)^2\times\frac{5}{8\pi}$
$= \frac{125}{2}$

Volume is increasing by $\frac{125}{2}\,cm^3/s$

Question two

(b) (i) $M = 100 + Ae^{-kt}$

$\Rightarrow Ae^{-kt} = M - 100$

$\frac{dM}{dt} = -k \cdot Ae^{-kt}$

$= -k(M-100)$

$= k(100 - M)$

(ii) $t = 0, \ M = 0$

$M = 100 + Ae^{0}$

$A = -100$

(iii) $M = 100 - 100e^{-kt}$

$t = 15, \ M = 40$

$40 = 100 - 100e^{-k \times 15}$

$-60 = -100e^{-k \times 15}$

$\frac{3}{5} = e^{-k \times 15}$

$\log_e \frac{3}{5} = -k \times 15$

$k = -\frac{1}{15} \log_e \frac{3}{5}$

$t = 30, \ M = ?$

$M = 100 - 100e^{-k \times 30}$

$= 64$

Question three

a) $\lim_{x \to 0} \frac{\sin \frac{x}{4}}{\frac{x}{4}} \times \frac{\frac{1}{4}}{3}$

$= 1 \times \frac{1}{12}$

$= \frac{1}{12}$

Question three

(b) $\cos x - \sin x \equiv R\cos(x + \alpha)$

$R\cos(x+\alpha) = R(\cos x \cos\alpha - \sin x \sin\alpha)$

$= R\cos x \cos\alpha - R\sin x \sin\alpha$

$1 = R\cos\alpha \quad (1)$

$-1 = -R\sin\alpha \quad (2)$

$\frac{(2)}{(1)} \quad \frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{1}$

$\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$(1)^2 + (2)^2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 1^2$

$R^2(\cos^2\alpha + \sin^2\alpha) = 2$

$R^2 = 2$

$R = \pm\sqrt{2}, \ R > 0$

$R = \sqrt{2}$

$\therefore \cos x - \sin x = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

(ii) $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\cos\left(x + \frac{\pi}{4}\right) = \frac{1}{2}$

$x + \frac{\pi}{4} = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{12}, \frac{17\pi}{12}$

Question three

(c) LHS $= \dfrac{\tan A}{\tan 2A - \tan A}$

$$= \frac{\tan A}{\frac{2\tan A}{1-\tan^2 A} - \tan A}$$

$$= \frac{\tan A}{\frac{2\tan A - \tan A(1-\tan^2 A)}{1-\tan^2 A}}$$

$$= \frac{\tan A}{\frac{2\tan A - \tan A + \tan^3 A}{1-\tan^2 A}}$$

$$= \frac{\tan A}{\frac{\tan A + \tan^3 A}{1-\tan^2 A}}$$

$$= \tan A \times \left(\frac{1-\tan^2 A}{\tan A(1+\tan^2 A)}\right)$$

$$= \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$= \frac{1 - \frac{\sin^2 A}{\cos^2 A}}{1 + \frac{\sin^2 A}{\cos^2 A}}$$

$$= \frac{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\frac{\cos^2 A + \sin^2 A}{\cos^2 A}}$$

$$= \frac{\cos^2 A - \sin^2 A}{1}$$

$$= \cos 2A$$

$$= \text{RHS}$$

Question four

(a) $\sin 105^\circ = \sin(60+45)^\circ$

$= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

(b) $V = \pi \int_0^{\frac{\pi}{3}} (\sin 3x)^2 \, dx$

$= \pi \int_0^{\frac{\pi}{3}} \frac{1}{2}(1 - \cos 6x) \, dx$

$= \frac{\pi}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{3}}$

$= \frac{\pi}{2}\left(\left(\frac{\pi}{3} - \frac{1}{6}\sin 2\pi\right) - \left(0 - \frac{1}{6}\sin 0\right)\right)$

$= \frac{\pi}{2}\left(\left(\frac{\pi}{3} - 0\right) - (0)\right)$

$= \frac{\pi^2}{6}$

Volume is $\frac{\pi^2}{6}$ units^2

$\cos 2\theta = 1 - 2\sin^2\theta$

let $\theta = 3x$.

$\cos 6x = 1 - 2\sin^2 3x$

$\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$

(c) $\frac{d}{dx}(x \sin 3x)$

$= x \times 3\cos 3x + \sin 3x \times 1$

$= 3x \cos 3x + \sin 3x$

$\int_0^{\frac{\pi}{2}} 3x \cos 3x \, dx = \int_0^{\frac{\pi}{2}} \frac{d}{dx}(x \sin 3x) - \sin 3x \, dx.$

$\int_0^{\frac{\pi}{2}} x \cos 3x \, dx = \frac{1}{3}\left(x \sin 3x + \frac{1}{3}\cos 3x\right)_0^{\frac{\pi}{2}}$

$= \frac{1}{3}\left(\left(\frac{\pi}{2}\sin\frac{3\pi}{2} + \frac{1}{3}\cos\frac{3\pi}{2}\right) - \left(0 + \frac{1}{3}\cos 0\right)\right)$

$= \frac{1}{3}\left(\left(-\frac{\pi}{2} + 0\right) - \left(\frac{1}{3}\right)\right)$

$= -\frac{\pi}{6} - \frac{1}{9}$

Question five

(a) $y = \tan 2x$.

$\frac{dy}{dx} = 2\sec^2 2x$.

$x = \frac{\pi}{6}$, $\frac{dy}{dx} = 2\sec^2 \frac{\pi}{3}$ $\qquad y = \tan \frac{\pi}{3}$

$= 2 \times \frac{1}{(\frac{1}{2})^2}$ $\qquad = \sqrt{3}$.

$= 8$

$y - \sqrt{3} = 8(x - \frac{\pi}{6})$

$y - \sqrt{3} = 8x - \frac{4\pi}{3}$.

(b) $y = -4x - 5$ $\qquad 3y = -6x + 7$

$m_1 = -4$ $\qquad m_2 = -2$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{-4 - -2}{1 + (-4)(-2)}\right|$$

$$\tan\theta = \left|\frac{-2}{9}\right|$$

$\theta = 12^\circ 32'$

(c) $\sin 2\theta + \cos\theta = 0$

$2\sin\theta\cos\theta + \cos\theta = 0$.

$\cos\theta(2\sin\theta + 1) = 0$

$\cos\theta = 0$ $\qquad 2\sin\theta = -1$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\qquad \sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$.

Question six

(a) $5\sin\theta - 2\cos\theta = 2$

$5\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) = 2.$

$5(2t) - 2(1-t^2) = 2(1+t^2)$

$10t - 2 + 2t^2 = 2 + 2t^2$

$10t = 4$

$t = \frac{2}{5}.$

$\tan\frac{\theta}{2} = \frac{2}{5}.$

$\left(\frac{\theta}{2} = 21^\circ 48'\right)$

$\theta = 43^\circ 36'$

Check 180°

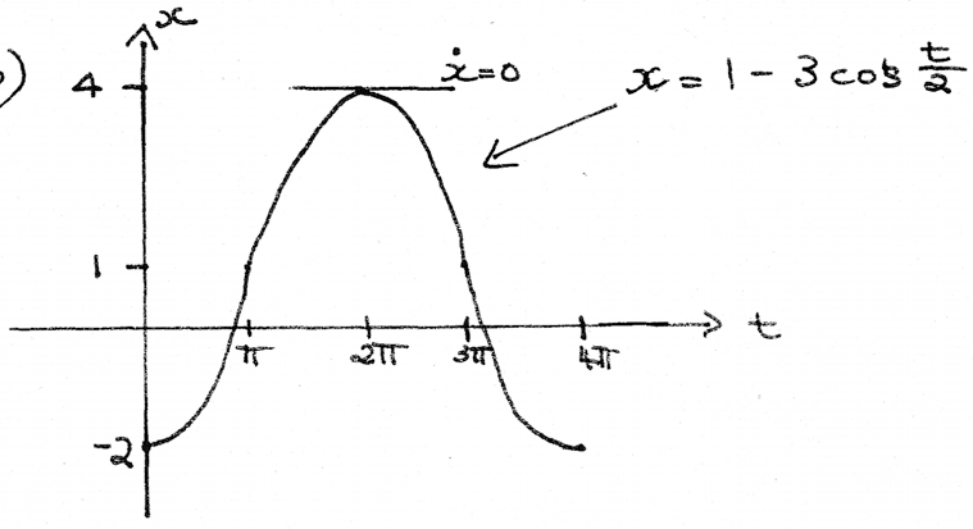
$5\sin 180^\circ - 2\cos 180^\circ = 0 - 2\times -1$

$= 2$

$= RHS.$

$\therefore$ Soln $\theta = 43^\circ 36', 180^\circ$

(b)



(ii) $\dot{x} = 0$, $t = 2\pi$ s $\quad x = 4$ m

(iii) max speed $t = \pi$s (or 3π)

$\dot{x} = \frac{3}{2}\sin\frac{t}{2} = \frac{3}{2}\sin\frac{\pi}{2} = \frac{3}{2}$

$\therefore$ max speed $= \frac{3}{2}$ m/s