NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 1 Assessment Task 2
Term 1, 2009

Name:_______________________________   Mathematics Class:_________

Time Allowed: 60 minutes + 2 minutes reading time

Available Marks: 51

Instructions:

- Questions are not of equal value. Part marks are indicated in each question.
- Start each question on a new page.
- Put your name on the top of each page.
- Attempt all seven questions.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Write on one side of the page only. Do not work in columns.
- Each question will be collected separately.
- If you do not attempt a question, you must submit a blank page with your name and the question number clearly displayed.

Question	1	2	3a	3b	4a	4b	5a	5b	6	7	Total	
PE3		/7		/2		/2	/4			/9	/24	
PE4									/8		/8	
HE2			/4								/4	
HE7	/7				/5			/3			/15	
											/51	%

Question 1 (7 marks) **Marks**

a) Find the sum of all the multiples of 4 between 1 and 999. **3**

b) Express the series 22 + 29 + 36 + … + 85 using sigma notation. **2**

c) Evaluate $\sum_{n=1}^{\infty} 5 \times \left(\frac{1}{4}\right)^{n-1}$ **2**

Question 2 (7 marks) **Start a new page.**

a) Two tangents are drawn from the point $R(6,-4)$ to the parabola $x^2 = 16y$.

i) State the equation of the chord of contact of the tangents from *R*. **1**

ii) Find the points of intersection of this chord with the parabola. **3**

iii) Show that the chord of contact is a focal chord. **1**

b) A car licence plate has 3 letters and 3 digits in that order.

i) How many different licence plates can be composed from odd numbers and vowels, without repetition? **1**

iii) A witness to a hit and run accident saw the first 2 letters and the last digit of the plate of the car involved. If the letters and digits can be repeated, how many license plates must be checked by the police to find the culprit? **1**

Please turn the page

Question 3 (6 marks) **Start a new page.** **Marks**

a) Use mathematical induction to prove that, for any positive integer *n*, **4**

$$(2)2^0+(3)2^1+(4)2^2+\ldots+(n+1)2^{n-1}=n\times 2^n$$

b) i) A wealthy investor, who owns 10 properties in Sydney, is preparing her will. In how many ways can the investor leave these properties to her three heirs? **1**

ii) The investor dies and 10 people attend the funeral. Each person shakes hands with every other attendee exactly once. How many handshakes are made? **1**

Question 4 (7 marks) **Start a new page.**

a) The first, fifth and thirteenth terms of an arithmetic series are the first three terms of a geometric series with a common ratio of 2. If the twenty-first term of the arithmetic series is 72, find:

i) the first term and the common difference of the arithmetic series **4**

ii) the sum of the first 10 terms of the arithmetic series. **1**

b) How many 3-digit numbers can be made from the digits 0, 3, 4, 5, 7 and 9? **2**

Question 5 (7 marks) **Start a new page.**

a) A radio station producer is scheduling a certain band's songs in the station's daily playlist. There are six slots available for their songs. In how many ways may the producer schedule the songs:

i) If the band has six different songs, each to be played once? **1**

ii) If the band has three different songs, each to be played twice? **2**

iii) If the band has two different songs, each to be played three times consecutively? **1**

b) The sum of *n* terms of an arithmetic series is given by the formula $S_n=4n+\frac{1}{2}n^2$.

Find:

i) an expression for S_{n-1} **1**

ii) a simple expression for the *n*th term, T_n. **2**

Please turn the page

Question 6 (8 marks) **Start a new page.** **Marks**

$P\left(6p, 3p^2\right)$ is a variable point moving on the parabola $x^2 = 12y$. S is the focus. SM is the perpendicular from the focus to the point M on the normal at P.

i) Show that the equation of the normal at the point P has the equation **2**

$$x + py = 3p^3 + 6p$$

ii) Show that M has the coordinates $\left(3p,\ 3p^2 + 3\right)$ **3**

iii) Find the locus of M and describe it geometrically. **3**

Question 7 (9 marks) **Start a new page.**

A stressed HSC student takes time out from her Mathematics study to play with the letters of the word *AGGRAVATE*.

i) How many distinct arrangements of the letters can she make? **2**

ii) What is the probability that in an arrangement of all the letters formed at random:

(α) the two *G*s are separated by at least one other letter? **2**

(β) the two *G*s are separated by exactly one letter, an *A*? **2**

iii) She intentionally discards all of the *A*s before placing the remaining letters in a bag. Four of the letters are drawn from the bag. What is the probability she does NOT draw a *G*? **3**

End of Paper

Year 12 Extension 1 Mathematics Task 2
SOLUTIONS

Question 1

a) Multiples of 4 form an A.S. with $a = 4$, $d = 4$, $l = 996$ and $n = 249$

$$S_{249} = \frac{n}{2}(a+l)$$
$$= \frac{249}{2}(4+996)$$
$$= 124500$$

b) Answers could include $\sum_{r=3}^{12}(7n+1)$ *or* $\sum_{r=1}^{10}(7n+15)$

c) $\sum_{n=1}^{\infty} 5\times\left(\frac{1}{4}\right)^{n-1} = 5+\frac{5}{4}+\frac{5}{16}+\ldots$

This forms a G.S. with $a = 5$, $r = \frac{1}{4}$

(since $|r|<1$ a limiting sum exists)

$$S_\infty = \frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$$

Question 2

a) i) Chord of contact given by $xx_1 = 2ay + 2ay_1$

Substituting $x_1 = 6,\ y_1 = -4,\ a = 4$

$6x = 2(4)y + 2(4)(-4)$

$\therefore 3x-4y+16=0$ is the equation of the chord of contact

ii)
$$x^2 = 16y \qquad (1)$$
$$3x-4y+16=0 \qquad (2)$$
$$16y = 12x+64 \qquad (2a)$$

(1) = (2a) $\quad x^2 = 12x+64$
$$x^2-12x-64=0$$
$$(x-16)(x+4)=0$$
$$x=16,\quad x=-4$$

From (1) when $x = 16$, $y = 16$ and when $x = -4$, $y = 1$

Points of intersection are $(16, 16)$ and $(-4, 1)$.

iii) Focus is $(0, 4)$. Substituting into chord of contact

$3x-4y+16=0$:

$LHS = 3(0)-4(4)+16 = 0 = RHS$

So chord of contact is a focal chord.

b) i) $5\times4\times3\times5\times4\times3 = 3600$

ii) Only one letter and 2 digits unaccounted for

Need to check $26\times10^2 = 2600$ plates

Question 3

a) *1. Prove true for n = 1*

LHS = $(2)2^0$ $\quad$ RHS = 1×2^1

= 2 $\quad$ = 2

$\therefore$ statement is true for $n = 1$

2. Let k be a value of n for which the statement is true

ie. $(2)2^0+(3)2^1+(4)2^2+\ldots+(k+1)2^{k-1} = k\times2^k$

3. Prove true for n = k + 1

rtp $(2)2^0+(3)2^1+(4)2^2+\ldots+(k+1)2^{k-1}+(k+2)2^k = (k+1)\times2^{k+1}$

LHS = $(2)2^0+(3)2^1+(4)2^2+\ldots+(k+1)2^{k-1}+(k+2)2^k$

$= k\times2^k+(k+2)2^k$ $\quad$ *(from 2)*

$= (k+k+2)2^k$

$= (k+1)\times2\times2^k$

$= (k+1)\times2^{k+1}$

= RHS

Thus by mathematical induction the statement is true for all $n\ge1$

b) i) $3^{10} = 59049$

ii) ${}^{10}C_2 = 45$

Question 4

a) i) 1st, 5th and 13th terms of an A.S. are a, $a + 4d$ & $a + 12d$ respectively.

These terms form a G.S., thus $\dfrac{a+4d}{a} = \dfrac{a+12d}{a+4d} = r$

From data, solve $\dfrac{a+4d}{a} = 2$

$$a + 4d = 2a$$

$$4d = a \qquad (1)$$

$T_{21} = a + 20d$

$72 = a + 5a$ using (1)

$\therefore a = 12$

Also from (1), $4d = 12 \ \therefore d = 3$

So the first term = 12 and the common difference is 3.

ii) For an A.S. $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$$\therefore S_{10} = \tfrac{10}{2}\left[2(12) + 9(3)\right] = 5 \times 53 = 159$$

b) only restriction is that a 3 digit number cannot start with 0

$\therefore 5 \times 6 \times 6 = 180$ numbers

Question 5

a) i) 6 distinct objects can be ordered 6! = 720 ways

ii) Consider this as 112233: 6 objects, 3 repetitions = $\dfrac{6!}{2!2!2!} = 90$ ways

Another option is to use combinations = $6C2 \times 4C2 = 90$ ways

iii) There are only two groups (111222) to arrange, so = 2 ways

b) i) $S_{n-1} = 4(n-1) + \frac{1}{2}(n-1)^2$

$$= \tfrac{1}{2}(n-1)\left[8 + n - 1\right]$$

$$= \tfrac{1}{2}(n-1)(n+7)$$

ii) $T_n = S_n - S_{n-1}$

$$= 4n + \tfrac{1}{2}n^2 - \tfrac{1}{2}(n-1)(n+7)$$

$$= \tfrac{1}{2}\left[8n + n^2 - (n^2 + 6n - 7)\right]$$

$$= \tfrac{1}{2}(2n + 7)$$

$$= n + \tfrac{7}{2}$$

Question 6

i) $y = \dfrac{x^2}{12}$

$\dfrac{dy}{dx} = \dfrac{2x}{12} = \dfrac{x}{6}$

at $x = 6p$,

$\dfrac{dy}{dx} = \dfrac{6p}{6} = p$ is the gradient of the tangent

$\therefore$ gradient of normal is $-\dfrac{1}{p}$

eqn of normal is given by

$$y - 3p^2 = -\frac{1}{p}(x - 6p)$$

$$py - 3p^3 = -x + 6p$$

$$x + py = 6p + 3p^3$$

ii) If SM is perpendicular to the normal from i), its gradient must be p since $m_1 m_2 = -1$ in perpendicular lines.

Since $a = 3$, focus S has coordinates (0, 3)

$\therefore$ eqn of SM is given by

$$y - 3 = p(x - 0)$$

$$y = px + 3 \qquad (1)$$

To find M, solve simultaneously with the eqn of the normal

$$x + py = 6p + 3p^3 \qquad (2)$$

(1)→(2)

$$x + p(px + 3) = 6p + 3p^3$$
$$x + p^2x + 3p = 6p + 3p^3$$
$$x(1 + p^2) = 3p + 3p^3$$
$$= 3p(1 + p^2)$$
$$\therefore x = 3p$$

→ (1)

$$y = p(3p) + 3$$
$$= 3p^2 + 3$$

$\therefore M$ has coordinates $\left(3p,\, 3p^2 + 3\right)$

iii)

$$x = 3p$$
$$\therefore p = \frac{x}{3} \qquad (A)$$
$$y = 3p^2 + 3 \qquad (B)$$

(A)→(B)

$$y = 3\left(\frac{x}{3}\right)^2 + 3$$
$$= \frac{x^2}{3} + 3$$
$$x^2 = 3(y - 3)$$

M is a parabola with vertex (0, 3) and focal length $\frac{3}{4}$.

<u>Question 8</u>

i) 9 letters: 3 A's, 2 G's

$$= \frac{9!}{3!2!} = 30240$$

ii) α) P(separated) = 1 – P(together)

First find no. of arrangements where G's are together:
Group as 1 element (note: no internal arrangements to consider)

$\therefore \frac{8!}{3!}$ = 6720 possible arrangements

$\therefore P$(separated) = $1 - \frac{6720}{30240} = \frac{7}{9}$

β) Group the "GAG" as one element. There are no internal arrangements but now only 2 A's to arrange outside this grouping so $\frac{7!}{2!} = 2520$ways.

$\therefore P$(GAG) = $\frac{2520}{30240} = \frac{1}{12}$

iii) There are now 6 letters: *GGRVTE*
In choosing 4 letters there are 3 cases:

1. No Gs selected = ${}^4C_4 = 1$ way
2. A single G selected = ${}^2C_1 \times {}^4C_3 = 8$ ways
3. Both Gs selected = ${}^2C_2 \times {}^4C_2 = 6$ ways

So P(no Gs) $= \frac{1}{15}$

ALTERNATIVELY

The simple probability way: Choose 4 cards:

$$\frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{15}$$