NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 1

Assessment Task 2 Term 1 2013

Name: ________________________ **Mathematics Class:** ____________

Student Number: ____________________

Time Allowed: **55 minutes + 2 minutes reading time**

Total Marks: **42**

Section I **(5 marks)**

- Attempt Questions 1 – 5
- Indicate your answer by colouring the appropriate circle on the answer sheet provided

Section II **(37 marks)**

- Attempt Questions 6-9
- Write your answers on the examination booklet provided
- Write on one side of the page only
- Do not work in columns
- Attempt all questions
- Show all necessary working
- Marks may be deducted for incomplete or poorly arranged work

Question	1 -4	5	6abc	6d	7	8a	8b	9a	9b	Total
HE2		/1		/3					/3	/7
HE6			/8			/4				/12
HE7	/4				/9		/4	/6		/23
										/42

SECTION I
5 marks
Attempt Questions 1 – 5
Use the multiple choice answer sheet for Questions 1 – 5.

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1 Given that $\log_a x = 2.8$ and $\log_a y = 4.3$, the value of $\log_a\left(\frac{y^2}{ax}\right)$ is given by the expression

(A) $2\times 4.3 - 2.8 + 1$

(B) $\frac{2\times 4.3}{2.8\times 1}$

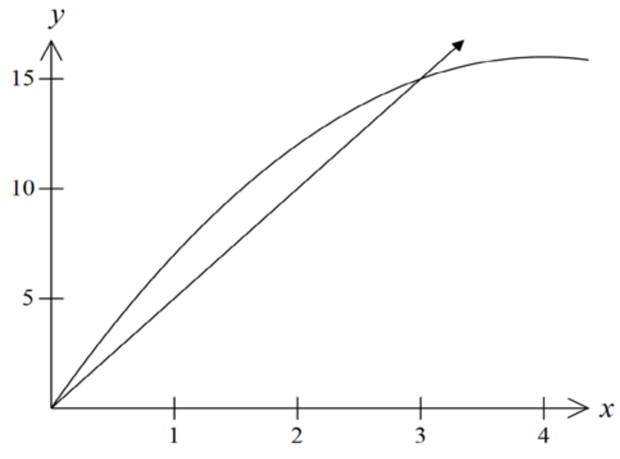
(C) $4.3^2 - 2.8\times 1$

(D) $2\times 4.3 - 2.8 - 1$

2 Let $k = 3^x$. Which expression is equal to $\log_3 k^2$?

(A) 3^{2x} (B) 3^{x^2} (C) $2x$ (D) x^2

3 The diagram below shows the graph of $y = 5x$ and $y = 8x - x^2$.
The graphs intersect at (3, 15).



Which integral will give the area enclosed between $y = 5x$ and $y = 8x - x^2$?

(A) $\int_0^3 \left(3x - x^2\right) dx$ (B) $\int_0^3 \left(x^2 - 3x\right) dx$

(C) $\int_0^3 \left(13x - x^2\right) dx$ (D) $\int_0^3 \left(x^2 - 13x\right) dx$

4 The graph of $y = f(x)$ has been drawn to scale for $0 \le x \le 2$.

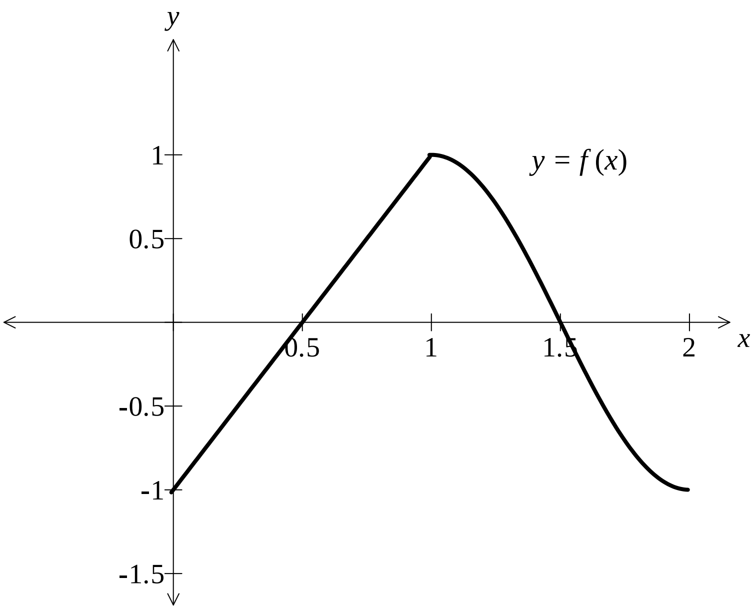
Which of the following has the greatest value?

(A) $\displaystyle\int_0^{0\cdot5} f(x)\,dx$

(B) $\displaystyle\int_0^{1} f(x)\,dx$

(C) $\displaystyle\int_0^{1\cdot5} f(x)\,dx$

(D) $\displaystyle\int_0^{2} f(x)\,dx$



5 The function $Q(n) = \dfrac{n}{6}(n+1)(4n+5)$ will sum the series

(A) $1\times3+2\times4+\ldots+n(n+2)$

(B) $1\times3+2\times5+\ldots+n(2n+1)$

(C) $1\times2+2\times3+\ldots+\dfrac{n}{2}(n+1)$

(D) $1\times2+2\times3+\ldots+n(n+1)$

SECTION II
40 marks
Attempt Questions 6 – 9
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 6 (11 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\displaystyle\int_1^9 \left(1+\sqrt{x}\right)\,dx$. **2**

(b) Find $\displaystyle\int x\sqrt{1-x^2}\;dx$ by using the substitution $u=1-x^2$. **3**

(c) Use the substitution $u=3-x$ to evaluate $\displaystyle\int_1^3 x\left(3-x\right)^4\,dx$. **3**

(d) Use mathematical induction to prove that **3**

$$\frac{1}{1\times 4}+\frac{1}{4\times 7}+\ldots\ldots+\frac{1}{\left(3n-2\right)\left(3n+1\right)}=\frac{n}{3n+1}$$

for all positive integers *n*.

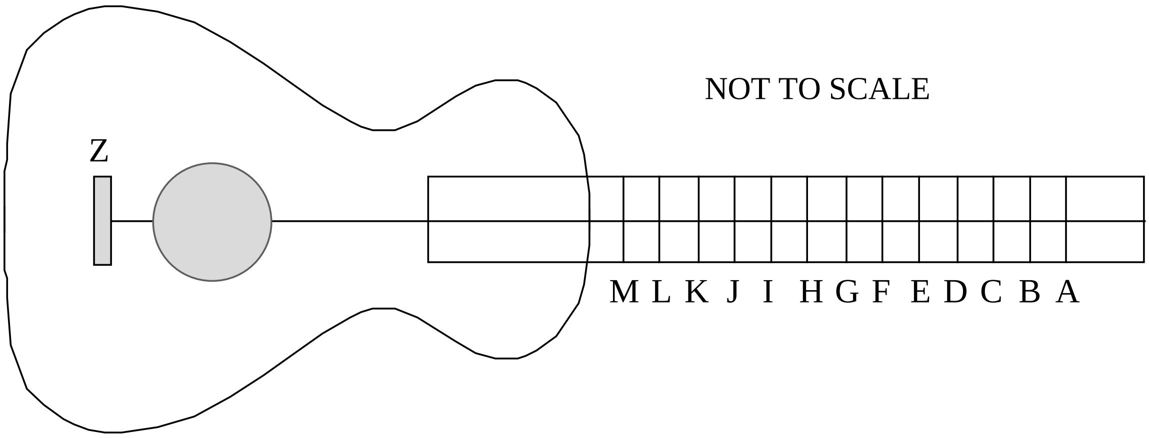
Question 7 (9 marks) Use a SEPARATE writing booklet.

(a) Find the least value of n such that $(0 \cdot 95)^n < 0 \cdot 5$, where n is an integer. **2**

(b) (i) Expand $\sum_{n=1}^{4} \left(1 - \frac{3^n - 2}{3^n}\right)$ and evaluate the sum. **2**

(ii) Find $\sum_{n=1}^{\infty} \left(1 - \frac{3^n - 2}{3^n}\right)$. **1**

(c) On the keyboard of a guitar, the mark M is the midpoint of A and Z.
The 13 marks lettered A to M are such that their distances from Z are terms in a geometric series.
The length AZ is 52 cm.



(i) Show that $r = \sqrt[12]{2}$. **2**

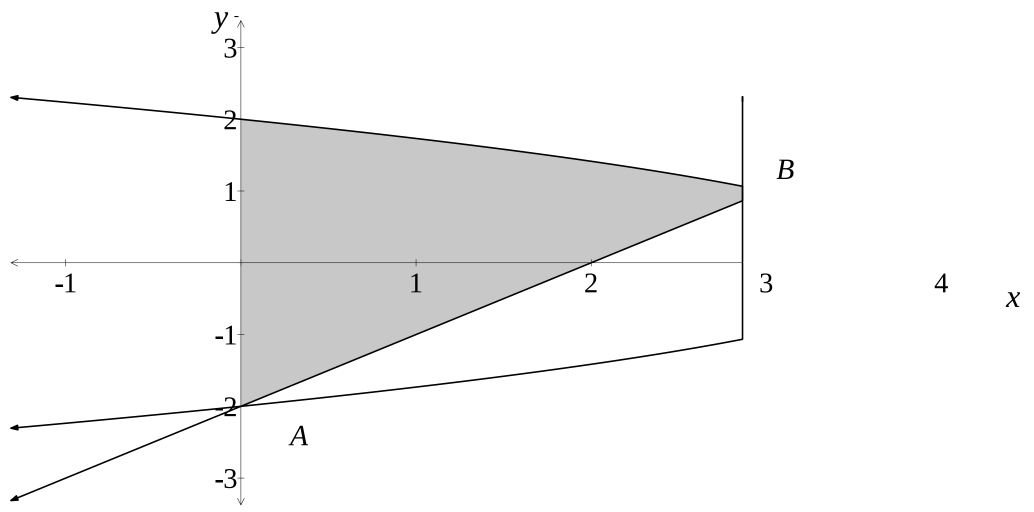
(ii) Find the distance FG correct to 1 decimal place. **2**

Question 8 (8 marks) Use a SEPARATE writing booklet.

(a) (i) Differentiate $x\sqrt{x+3}$, leaving your answer as a simple fraction. **2**

(ii) Hence find $\displaystyle\int \frac{x+2}{\sqrt{x+3}}\,dx$. **2**

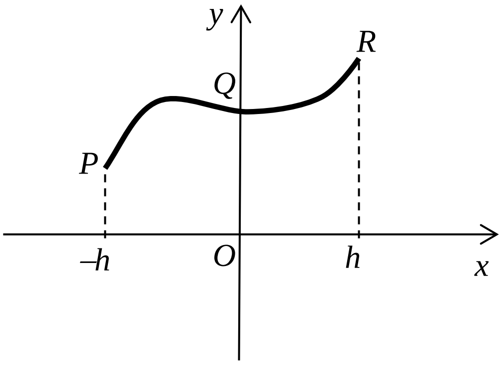
(b) The line $y = x - 2$ meets the sideways parabola $y^2 = 4 - x$ at two points $A(0,-2)$ and $B(3,1)$ as shown.



The shaded region is rotated about the y-axis.
Calculate the volume of the solid of revolution. **4**

Question 9 (9 marks) Use a SEPARATE writing booklet.

(a) Consider three points on the curve $y = ax^3 + bx^2 + cx + d$ with coordinates, $P(-h, y_0)$, $Q(0, y_1)$ and $R(h, y_2)$.



(i) Find expressions for y_0, y_1 and y_2 in terms of *h, a, b ,c* and *d*. **2**

(ii) Hence write down an expression for $y_0 + 4y_1 + y_2$. **1**

(iii) Find a simplified expression for $\int_{-h}^{h} \left(ax^3 + bx^2 + cx + d\right) dx$, using calculus. **2**

(iv) Hence deduce that Simpsons Rule gives an exact value for the area bound by a cubic and the *x*-axis. **1**

(b) Use mathematical induction to prove that $5^n > 3^n + 4^n$ for all positive integers $n \geq 3$. **3**

End of paper

Question: Name:

Start here :

Solutions Ext 1 Task 2 2013

Multiple Choice

1. $\log \frac{y^2}{ax} = 2\log_a y - (\log_a a + \log_a x)$

$= 2 \times 4 \cdot 3 - 1 - 2 \cdot 8$ (D)

2. $k = 3^x$

$\therefore \log_3 k = x$

$\therefore \log_3 k^2 = 2\log_3 k$

$= 2x$ (C)

3. Area $= \int_0^3 (y_2 - y_1)\, dx$

$= \int_0^3 (8x - x^2) - 5x\, dx$

$= \int_0^3 3x - x^2\, dx$ (A)

4: $\int_0^{1.5} f(x)\, dx$ contains the greatest proportion above the x-axis. The value will be higher than any of the other integrals

(C)

5. $Q(n) = \frac{n}{6}(n+1)(4n+5)$ where $Q(n)$ is the sum

Let $n=1$ $Q(1) = 3$ $\Rightarrow$ either (A) or (B) could be right.

$n=2$ $Q(2) = \frac{1}{3} \times 3 \times 13$

$= 13$

$\therefore$ (B) is correct since $1 \times 3 + 2 \times 5 = 13$

Question: Name: Class: 12

Start here :

Question 6

a) $\int_1^9 (1+\sqrt{x})\ dx = \int_1^9 1 + x^{1/2}\ dx$

$= \left[x + \frac{2}{3}x^{3/2}\right]_1^9$

$= (9 + 18) - (1 + \frac{2}{3})$

$= 25\frac{1}{3}$

b) $u = 1-x^2$

$\frac{du}{dx} = -2x$

$-\frac{1}{2}du = x\,dx$

$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int \sqrt{u}\ du$

$= -\frac{1}{2} \times \frac{2u^{3/2}}{3}$

$= -\frac{1}{3}u^{3/2} + c$

$= -\frac{1}{3}(1-x^2)\sqrt{1-x^2} + C$

c) $u = 3-x \implies x = 3-u$

$du = -dx$

When $x = 3$, $u = 0$

When $x = 1$, $u = 2$

$\therefore \int_1^3 x(3-x)^4\,dx = -\int_2^0 (3-u)\,u^4\,du$

$= \int_0^2 3u^4 - u^5\ du$ } reversing order of limits

Question:______________ Student Number: ___

Start here :

$$= \left[\frac{3u^5}{5} - \frac{u^6}{6} \right]_0^2$$

$$= 8\,{}^{8}/_{15} \text{ or } \frac{128}{15}$$

d) $\frac{1}{1\times4} + \frac{1}{4\times7} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$ for $n \in Z^+$

Test result for $n=1$: $LHS = \frac{1}{1\times4} = \frac{1}{4}$

$RHS = \frac{1}{4}$

$\therefore$ Result is true for $n=1$

Assume result holds for $n=k$

ie assume that $\frac{1}{1\times4} + \frac{1}{4\times7} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}$

We wish to show true for $n=k+1$

ie show that $\frac{1}{1\times4} + \frac{1}{4\times7} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)} = \frac{k+1}{3k+4}$

$LHS = \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$ by assumption

$$= \frac{k(3k+4) + 1}{(3k+1)(3k+4)}$$

$$= \frac{3k^2 + 4k + 1}{(3k+1)(3k+4)}$$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$= \frac{k+1}{3k+4}$$

$\therefore$ By Mathematical Induction the result is proved true

Question: Name: Class: 12

Start here :

Q7 a) $(0.95)^n < 0.5$

$n = 14$ by trial and error on calc.

Alternatively, take logs of both sides:

$n \log 0.95 < \log 0.5$

$n > \dfrac{\log 0.5}{\log 0.95}$ sign reverses because $\log 0.95$ is negative

$n > 13.5$

$\therefore n = 14$ is the least value of n.

b) (i) $\displaystyle\sum_{n=1}^{4} 1 - \frac{3^n - 2}{3^n} = \sum_{n=1}^{4} \frac{3^n - 3^n + 2}{3^n}$ } simplify for ease of working

$\displaystyle = \sum_{n=1}^{4} \frac{2}{3^n}$

$= \dfrac{2}{3} + \dfrac{2}{9} + \dfrac{2}{27} + \dfrac{2}{81}$

$= \dfrac{79}{81}$

(ii) $a = \dfrac{2}{3}$ and $r = \dfrac{1}{3}$

$S = \dfrac{a}{1-r}$

$= 1$

Question:______________ Student Number: ______________

Start here :

c) All distances are measured from Z

ie $AZ = 52$ cm and $MZ = 26$ cm

Let $a = 26$ (MZ) in geometric series

then $\frac{LZ}{MZ} = r$ where r is the ratio

$LZ = 26r$

$\therefore AZ = 26r^{12}$ since there are twelve letters after M (L, K, J, I, H, G, F, E, D, C, B, A)

$52 = 26r^{12}$

$r^{12} = 2$

$r = \sqrt[12]{2}$

(ii) $FG = FZ - GZ$

$= 26r^7 - 26r^6$

$= 26\left(2^{7/12} - 2^{6/12}\right)$

$= 2.186$

$= 2.2$ cm

Question: Name: Class: 12

Start here :

Question 8

a) $\frac{d}{dx}\left[x\sqrt{x+3}\right] = \sqrt{x+3} + x \times \frac{1}{2}(x+3)^{-1/2} \times 1$ product rule

$= \sqrt{x+3} + \frac{x}{2\sqrt{x+3}}$

$= \frac{2(x+3)+x}{2\sqrt{x+3}}$ simplify to make work easier

$= \frac{3x+6}{2\sqrt{x+3}}$

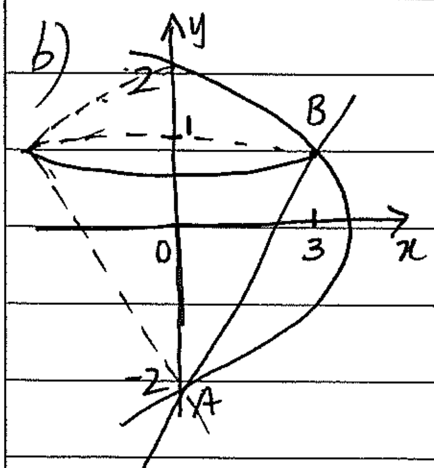
$= \frac{3(x+2)}{2\sqrt{x+3}}$

not in marking scheme for (i)
Helpful for (ii)

(ii) $\int \frac{x+2}{\sqrt{x+3}}\,dx = \frac{2}{3}\int \frac{3x+6}{2\sqrt{x+3}}\,dx$

$= \frac{2}{3}x\sqrt{x+3} + C$

b)



Volume is considered in two parts:

V_1 rotating the line about y axis betw $y=-2, 1$

V_2 rotating the curve about y axis betw $y=1, 2$

$V = \pi\int_{-2}^{1}(y^2+4y+4)\,dy + \pi\int_{1}^{2} 16-8y^2+y^4\,dy$

$= \pi\left[\frac{y^3}{3}+2y^2+4y\right]_{-2}^{1} + \pi\left[16y-\frac{8y^3}{3}+\frac{y^5}{5}\right]_{1}^{2}$

$= 9\pi + 3\frac{8}{15}\pi$

$= 12\frac{8}{15}\pi\,u^3$ or $\frac{188\pi}{15}$ units^3

Vol of cone
$= \frac{\pi}{3}r^2h$
$= \frac{\pi}{3}\times 3^2 \times 3$
$= 9\pi$

Question 9

(i) $y_0 = a(-h)^3 + b(-h)^2 + c(-h) + d$

$= -ah^3 + bh^2 \quad ch + d$

$y_1 = d$

$y_2 = ah^3 + bh^2 + ch + d$

(ii) $y_0 + 4y_1 + y_2 = 2bh^2 + 4d + 2d$ since odd powers add to give zero

$= 2bh^2 + 6d$

(iii) $\int_{-h}^{h} (ax^3 + bx^2 + cx + d)\, dx$

$= \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-h}^{h}$

$= \left(\frac{ah^4}{4} + \frac{bh^3}{3} + \frac{ch^2}{2} + dh\right) - \left(\frac{ah^4}{4} - \frac{bh^3}{3} + \frac{ch^2}{2} - dh\right)$

$= \frac{2bh^3}{3} + 2dh$

(iv) $\frac{h}{3}(y_0 + 4y_1 + y_2) \doteqdot$ area under a curve by Simpsons rule

$\frac{h}{3}(y_0 + 4y_1 + y_2) = \frac{h}{3}(2bh^2 + 6d)$ from (ii)

$= \frac{2bh^3}{3} + 2hd$

$= \int_{-h}^{h} (ax^3 + bx^2 + cx + d)\, dx$

$\therefore$ Simpsons rule gives an exact value for the area bound by a cubic and x-axis

Q9 b) Show $5^n > 3^n + 4^n$ for $n \geq 3$, $n \in Z^+$

Test if true for $n=1$ $\quad LHS = 5^3 = 125$

$RHS = 3^3 + 4^3$

$= 91$

$\therefore$ Result is true for $n=3$

Assume result is true for $n=k$ ie assume $5^k > 3^k + 4^k$

We wish to show that $5^{k+1} > 3^{k+1} + 4^{k+1}$ is true

$$\begin{aligned} LHS &= 5 \times 5^k \\ &> 5(3^k + 4^k) \text{ by assumption} \\ &= 5 \times 3^k + 5 \times 4^k \\ &> 3 \times 3^k + 4 \times 4^k \\ &= 3^{k+1} + 4^{k+1} \end{aligned}$$

$\therefore 5^{k+1} > 3^{k+1} + 4^{k+1}$

$\therefore$ By mathematical Induction result is true for all $n \geq 3$, $n \in Z^+$

Alternatively consider $5^{k+1} - (3^{k+1} + 4^{k+1})$

$$\begin{aligned} &> 5(3^k + 4^k) - 3.3^k - 4 \times 4^k \\ &= 2 \times 3^k + 2 \times 4^k \\ &> 0 \end{aligned}$$

$\therefore 5^{k+1} > 3^{k+1} + 4^{k+1}$