June 2002 Ext 1
(HY)

3U ASS
2002 PLC

Question 1 (11 marks) Start a new page. **Marks:**

(a) Find $\frac{d}{dx} \tan^{-1} \frac{x}{2}$ **1**

(b) Find $\int \frac{1}{\sqrt{2-x^2}}\ dx$ **1**

(c) i) Sketch the graph of $y = 2\cos^{-1}(\frac{x}{3})$

ii) State the domain and range of this function. **3**

(d) Without the aid of calculus, sketch the graph of the following polynomial function, clearly indicating all intercepts with the axes.

$$f(x) = -x(x+2)^2(x-1)$$ **2**

(e) Use the substitution $u = x^2 - 1$ to evaluate:

$$\int_1^2 2x(x^2-1)^4\, dx$$ **4**

Question 2 (11 marks) Start a new page. **Marks:**

(a) A polynomial $P(x) = x^4 - 3x^3 + ax^2 + bx - 6$ has a remainder of 8 when divided by $(x+1)$. Also, $(x-3)$ is a factor of $P(x)$. Find a and b.

3

(b) Consider the function $f(x) = 1 + \sqrt{x+1}$

i) State the largest domain of $f(x)$ for which $f^{-1}(x)$ is defined.

ii) Find $f^{-1}(x)$, stating its domain and range.

ii) Sketch both $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

iii) Find the point where $y = f(x)$ and $y = f^{-1}(x)$ intersect.

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Question 3 (12 marks)

(a) Find the general solution of $\sqrt{2}\sin 2\theta + 1 = 0$ **2**

(b) i) Show that $e^{-x} - \cos x = 0$ has a root near $x = 1.31$.

ii) Use one step of Newton's method to find a closer approximation to the root. Answer to 3 decimal places.

4

c) Joshua wishes to invest a regular amount ($\$M$) at the beginning of each quarter so that after five years he will have \$20 000. Interest is to be paid quarterly on the balance at 9% p.a.

i) Show that after n quarters the balance of his investment is:

$$\frac{\$M(1+R)((1+R)^n - 1)}{R}$$

where R is the quarterly interest rate.

ii) Hence or otherwise calculate M, the amount of each quarterly investment, to the nearest cent. **6**

Question 4 (10 marks) Start a new page. **Marks:**

(a) Find all the real roots of $2x^3 - 3x^2 - 11x + 6 = 0$. **4**

(b) Rhys borrows \$94 000 to purchase his house. The loan is over 25 years at 10% p.a. interest, compounded on the balance owing before each monthly repayment. He used the following formula to calculate the value of his monthly repayments:

$$M = \frac{P(1+R)^n R}{((1+R)^n - 1)}$$

where P is the amount of the loan,
R is the monthly interest rate,
n is the number of months.

i) Find the value of his monthly repayments, to the nearest cent.

ii) If Rhys decides to pay \$1092 per month off his loan instead, how many years and months does he save on his loan?

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Yr12 Extension 1 June 2002

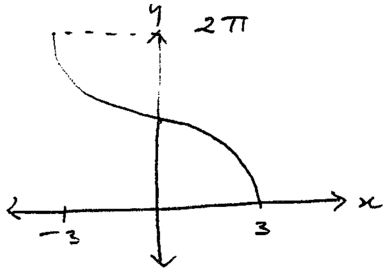
3U ASR
2002 PLC

Question 1.

a) $\frac{d}{dx} \tan^{-1}\frac{x}{2} = \frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{1}{2} = \frac{2}{4+x^2}$

b) $\int \frac{1}{\sqrt{2-x^2}}\,dx = \sin^{-1}\frac{x}{\sqrt{2}} + c$

c) i)



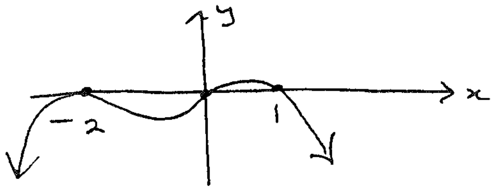
ii) Domain: $-1 \le \frac{x}{3} \le 1$

$\therefore$ $\underline{-3 \le x \le 3}$

Range $0 \le \cos^{-1}\frac{x}{3} \le \pi$

$\therefore$ $\underline{0 \le 2\cos^{-1}\frac{x}{3} \le 2\pi}$

d)



e) $du/dx = 2x$

When $x=1$ $u=0$

$x=2$ $u=3$

$\therefore \int_0^3 u^4\,du = \left[\frac{u^5}{5}\right]_0^3 = \underline{\frac{243}{5}}$

Question 2.

a) $P(-1) = 1+3+a-b-6 = 8$ ①

$P(3) = 81-81+9a+3b-6 = 0$ ②

① $a-b=10$ $\therefore 4a = 12$

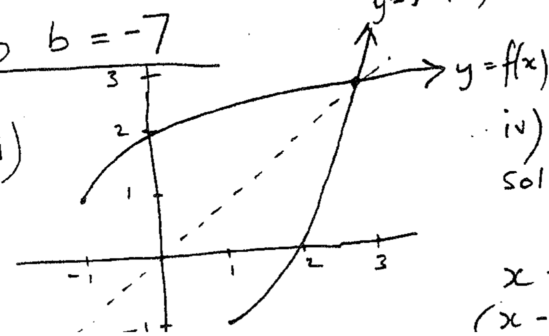
② $3a+b=2$ $\Rightarrow a=3,\ b=-7$

b) i) Domain $x \ge -1$

ii) $x = 1+\sqrt{y+1}$

$(x-1)^2 = y+1$

iii)



iv) $(3,3)$

solve $y=x$

$y = 1+\sqrt{x+1}$

$x = 1+\sqrt{x+1}$

$(x-1)^2 = x+1$

Question 3

a) $\sqrt{2}\sin 2\theta + 1 = 0$

$\sin 2\theta = -1/\sqrt{2}$

$2\theta = n\pi + (-1)^n \sin^{-1}(-1/\sqrt{2})$

$\theta = \frac{n\pi}{2} + (-1)^{n+1}\pi/8$

b) i) $f(1.29) = e^{-1.29} - \cos 1.29 = -0.0019$

$f(1.32) = e^{-1.32} - \cos 1.32 = 0.019$

Change in sign $\therefore$ root between.

ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ $\quad$ $f(x) = e^{-x} - \cos x$

$f'(x) = -e^{-x} + \sin x$

$\therefore x_2 = 1.31 - \frac{e^{-1.31} - \cos 1.31}{(-e^{-1.31} + \sin 1.31)}$

$= 1.293$

c) Let A_n = amount of investment after n quarters.

$R = \frac{9\%}{4} = 2.25\%$

$A_1 = \$M(1+R)$

$A_2 = \$M(1+R)(1+R) + \$M(1+R)$

$A_n = \$M(1+R)^n + \$M(1+R)^{n-1} + \ldots + \$M(1+R)$

$= \frac{\$M(1+R)\left[(1+R)^n - 1\right]}{1+R-1}$ $\quad$ as required

ii) After $5 \times 4 = 20$ quarters have \$20,000

$\therefore \$20,000 = \$M(1.0225)\frac{(1.0225^{20} - 1)}{0.0225}$

$\$M = \$785.17$

a) Let $P(x) = 2x^3 - 3x^2 - 11x + 6$

$P(3) = 0 \quad \therefore x - 3$ is a factor

$$\begin{array}{r}
(x-3)\overline{)2x^3 - 3x^2 - 11x + 6} \\
2x^3 - 6x^2 \\
\hline
3x^2 - 11x \\
3x^2 - 9x \\
-2x + 6 \\
-2x + 6 \\
\hline
0
\end{array}$$

$\therefore P(x) = (x-3)(2x^2 + 3x - 2)$

$= (x-3)(2x-1)(x+2)$

$= 0$ when $x = -2, 1/2, 3$.

b) i) $M = \frac{94000\left(1 + \frac{5}{600}\right)^{300} \cdot 5/600}{\left(1 + \frac{5}{600}\right)^{300} - 1} = \$854.18$

$R = \frac{10}{12}\% = \frac{5}{600}$

$n = 25 \times 12 = 300$ months

ii) Let $M = \$1092$ $\quad$ $\$1092 = \frac{94000(1 + 5/600)^n \times 5/600}{(1+5/600)^n - 1}$

$\Rightarrow 1092 \times (1 + 5/600)^n - 1092 = 94000(1+5/600)^n \times 5/600$

$\Rightarrow (1092 - 94000 \times 5/600)(1 + 5/600)^n = 1092$

$\Rightarrow n \ln(1 + 5/600) = \ln\left(\frac{1092}{1092 - \frac{2350}{3}}\right)$

$n = \frac{1.2635\ldots}{0.00829\ldots}$

$n = 152.25$

$= 12$ yrs 9 mths.

$\therefore$ He saved $\underline{12 \text{ yrs } 3 \text{ months}}$.