Marks

Question 1

(a) Differentiate:

(i) e^{3x-1} **1**

(ii) $\log_e (2x-1)$ **1**

(b) Evaluate $\lim_{x\to 0} \dfrac{\sin 3x}{x}$ **1**

(c) Find the remainder when the polynomial $P(x) = x^3 - 2x$ is divided by $x + 1$ **2**

(d) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}}$ **2**

(e) Find the equation of the tangent to the curve $y = e^{2x} - 3x$ at the point (0,1) **2**

(f) A(-2, -5) and B(1, 4) are two points. Find the acute angle ϑ between the line AB and the line $x + 2y + 1 = 0$, giving the answer correct to the nearest minute. **3**

Marks

Question 2
Start a new page

(a) Given $\log_a x = 0.64$ and $\log_a y = 0.04$ find $\log_a(\frac{x}{y})$ **1**

(b) Q(-1, 4) and R(x, y) are two points. The point P(14, -6) divides the interval QR externally in the ration 5:3. Find the coordinates of R. **2**

(c)

(i) Sketch $y = e^x - 1$ **2**

(ii) Find the exact volume of the solid of revolution formed when the curve $y = e^x - 1$ is rotated about the x-axis from $x = 0$ to $x = 1$ **3**

(d)

(i) Show that the function $x^3 - 2x - 5 = 0$ has one root which lies between 2 and 2.2 **1**

(ii) Using $x = 2.1$ as an approximation to a root of $x^3 - 2x - 5 = 0$, find a better approximation, correct to 2 decimal places, using Newton's method once. **3**

Marks

Question 3
Start a new page

(a) Find $\int \frac{x}{x^2+5}\,dx$ **2**

(b) Write the expansion of sin (A + B).
Hence, or otherwise, find the exact value of sin 75° **3**

(c) Solve for x: $\frac{4}{x+1} \le 3$ **3**

(d)

(i) Show that $(x + 1)$ is a factor of $f(x) = 2x^3 + 7x^2 - 7x - 12$ **1**

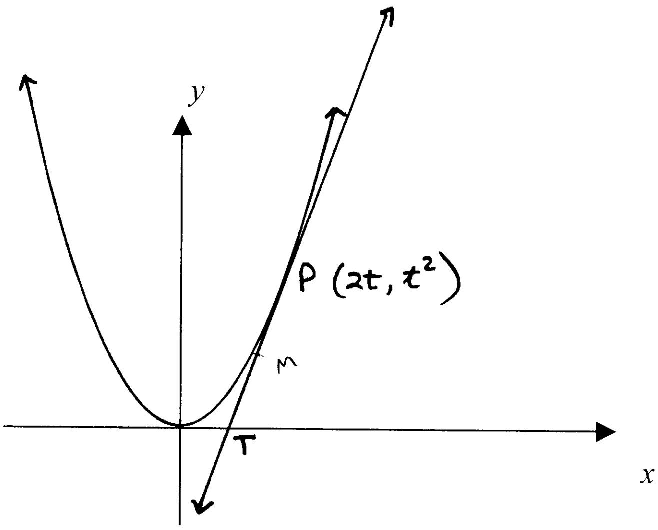
(ii) Find all roots of $f(x) = 2x^3 + 7x^2 - 7x - 12$ and hence sketch the curve. **3**

Marks

Question 4
Start a new page

(a) If α, β, γ are the roots of the equation $3x^3 - 6x^2 + x + 2 = 0$ find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **3**

(b)



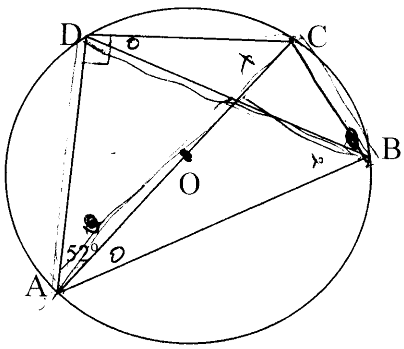
P $(2t, t^2)$ is a variable point which moves on the parabola $x^2 = 4y$. The tangent to the parabola at P cuts the x-axis at T. M is the midpoint of PT.

(i) Show that the tangent PT has equation $tx - y - t^2 = 0$ **2**

(ii) Show that M has coordinates $\left(\dfrac{3t}{2}, \dfrac{t^2}{2}\right)$ **2**

(iii) Hence find the Cartesian equation of the locus of M as P moves on the parabola **2**

(c)



not to scale

Circle, centre O, has diameter AC. $\angle DAC = 52^\circ$

(i) Explain why $\angle ADC = 90^\circ$ **1**

(ii) Find $\angle DBA$, giving reasons for your answer. **2**

Marks

Question 5
Start a new page

(a) Stephanie invests \$150 at the start of each month into a superannuation fund. The interest is compounded monthly at a rate of 3% p.a. The first \$150 is invested at the beginning of January 2005 and the last is invested at ~~the beginning~~ of December 2010. Calculate, to the nearest dollar:

(i) The amount to which the January 2005 investment will have grown by the end of 2010. **2**

(ii) The amount to which the total will have grown by the end of 2010. **3**

(b) Use mathematical induction to prove that: **3**

$$\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\ldots\ldots+\frac{1}{(n+1)(n+2)}=\frac{n}{2(n+2)}$$

(c) Consider the function $f(x) = 2\sin^{-1}x$

(i) Sketch the graph of $f(x) = 2\sin^{-1}x$ **1**

(ii) Find the exact value of $f(\frac{1}{\sqrt{2}})$ **1**

(iii) Find the equation of the tangent to the curve at the point where $x = \frac{1}{\sqrt{2}}$. **2**

Marks

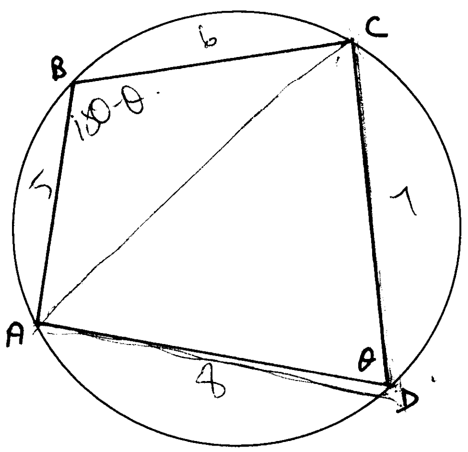
Question 6
Start a new page

(a) Find $\dfrac{d}{dx}\cos^{-1} 2x^3$ **2**

(b) If $t = \tan \dfrac{1}{2}\theta$, prove $\cos\theta\,(\tan\theta - \tan\dfrac{1}{2}\theta) = \tan\dfrac{1}{2}\theta$ **3**

(c) Show that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{8} - \frac{1}{4}$ **3**

(d)



not to scale

A, B, C, D are points on the circumference of a circle.

AB = 5cm, BC = 6cm, DC = 7cm, AD = 8cm

(i) If $\angle$ ADC = θ, explain why $\angle$ ABC = $180° - \theta$ **1**

(ii) By drawing the diagonal AC, or otherwi-

$\cos \angle ADC = \dfrac{13}{43}$ **3**

Marks

Question 7
Start a new page

(a) Using the substitution $u = e^x$, find

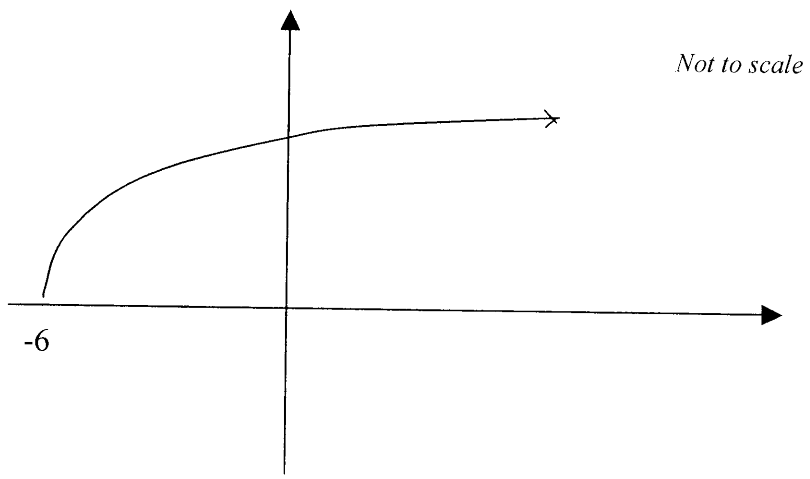
$$\int \frac{e^x}{1+e^{2x}} dx$$ **2**

(b)

(i) Express $\sqrt{3}\sin\vartheta - \cos\vartheta$ in the form $R\sin(\vartheta - \alpha)$, where α is in radians. **2**

(ii) Hence, or otherwise, find all angles ϑ, where $0 \le \vartheta \le 2\pi$, for which $\sqrt{3}\sin\vartheta - \cos\vartheta = 1$ **2**

(c) The graph of $f(x) = \sqrt{x+6}$ for $x \ge -6$ is shown in the diagram.



(i) Find the inverse function $f^{-1}(x)$ **1**

(ii) On the same diagram sketch the graphs of $y = f(x)$, the line $y = x$, and $y = f^{-1}(x)$. Show clearly the intercepts on the coordinate axes. **2**

(iii) What is the domain of $f^{-1}(x)$ **1**

(iv) Show that the x coordinate of any points of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfy the equation $x^2 - x - 6 = 0$.
Hence find any points of intersection of the two graphs. **2**

END OF EXAMINATION

3 unit

QUESTION 1

a) i) $y = e^{3x-1}$

$y' = 3e^{3x-1}$

ii) $y = \log_e(2x-1)$

$y' = \frac{2}{2x-1}$

b) $\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{3}{1}$

$= 3$

c) $P(x) = x^3 - 2x$

$P(-1) = (-1)^3 - 2(-1)$

$= -1 + 2$

$= 1$

d) $\int_0^{\sqrt3} \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt3}$

$= \sin^{-1}\frac{\sqrt3}{2} - \sin^{-1} 0$

$= \frac{\pi}{3} - 0$

$= \frac{\pi}{3}$

e) $y = e^{2x} - 3x$

$y' = 2e^{2x} - 3$

at $x = 0$, $y' = 2e^0 - 3$

$= 2 - 3$

$= -1$

$y - y_1 = m(x - x_1)$

$y - 1 = -1(x - 0)$

$y - 1 = -x$

$x + y - 1 = 0$

f) $A(-2,-5)$ $B(1,4)$

$m = \frac{4+5}{1+2}$

$= \frac{9}{3}$

$= 3$

$x + 2y + 1 = 10$

$2y = -x - 1$

$y = -\frac{1}{2}x - \frac{1}{2}$

$\therefore m = -\frac{1}{2}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{3 + \frac{1}{2}}{1 + 3 \times -\frac{1}{2}}\right|$

$= 7$

$\theta = 81° 52'$

QUESTION 2

a) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

$= 0.64 - 0.04$

$= 0.6$

b) $Q(-1,4)$ $R(x,y)$ $P(14,-6)$ $5:3$

$14 = \frac{5x - 3(-1)}{2}$ $-6 = \frac{5y - 3(4)}{2}$

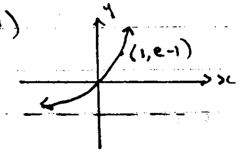
$28 = 5x + 3$ $-12 = 5y - 12$

$25 = 5x$ $0 = 5y$

$x = 5$ $y = 0$

$\therefore R(5,0)$

c) i)



ii) $V_{x\,axis} = \pi\int_a^b y^2\,dx$

$= \pi\int_0^1 (e^x - 1)^2\,dx$

$= \pi\int_0^1 e^{2x} - 2e^x + 1\,dx$

$= \pi\left[\frac{e^{2x}}{2} - 2e^x + x\right]_0^1$

$= \pi\left[\left(\frac{e^2}{2} - 2e^1 + 1\right) - \left(\frac{e^0}{2} - 2e^0 + 0\right)\right]$

$= \pi\left(\frac{e^2}{2} - 2e + 2\frac{1}{2}\right)$ units^3

d) i) $f(2) = (2)^3 - 2(2) - 5$

$= -1$

$f(2.2) = (2.2)^3 - 2(2.2) - 5$

$= 1.248$

Since one result is positive & the other negative, the root lies between 2 and 2.2.

ii) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(2.1) = (2.1)^3 - 2(2.1) - 5$

$= 0.061$

$f'(x) = 3x^2 - 2$

$f'(2.1) = 3(2.1)^2 - 2$

$= 11.23$

$x_1 = 2.1 - \frac{0.061}{11.23}$

$= 2.0945681$

$= 2.09$ to 2 dec pl

QUESTION 3

a) $\int \frac{x}{x^2+5}\,dx = \frac{1}{2}\int\frac{2x}{x^2+5}\,dx$

$= \frac{1}{2}\ln(x^2+5) + c$

b) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin 75 = \sin(45+30) = \sin 45\cos 30 + \cos 45 \sin 30$

$= \frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2} + \frac{1}{\sqrt2}\cdot\frac{1}{2}$

$= \frac{\sqrt3 + 1}{2\sqrt2}$

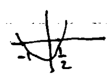
c) $\frac{4}{x+1} \times (x+1)^2 \le 3(x+1)^2$ $x \ne -1$

$4(x+1) \le 3(x+1)^2$

$0 \le 3(x+1)^2 - 4(x+1)$

$0 \le (x+1)(3(x+1) - 4)$

$(x+1)(3x-1) \ge 0$



$x \le -1$ and $x \ge \frac{1}{3}$

but $x \ne -1$

$\therefore x < -1$, $x \ge \frac{1}{3}$

d) i) $f(x) = 2x^3 + 7x^2 - 7x - 12$

$f(-1) = 2(-1)^3 + 7(-1)^2 - 7(-1) - 12$

$= -2 + 7 + 7 - 12$

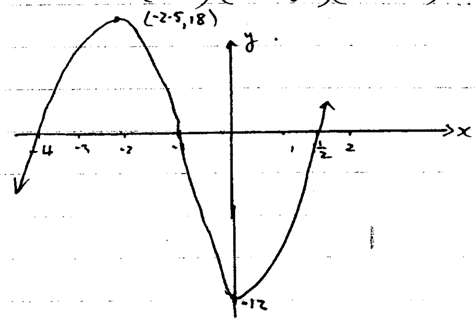
$= 0$

Since the remainder is 0, $(x+1)$ is a fact

ii)
$$
\begin{array}{r}
2x^2 + 5x - 12 \\
x+1 \overline{)\,2x^3 + 7x^2 - 7x - 12} \\
2x^3 + 2x^2 \\
\hline
5x^2 - 7x \\
5x^2 + 5x \\
\hline
-12x - 12 \\
-12x - 12 \\
\hline
\end{array}
$$

$\therefore f(x) = (x+1)(2x^2 + 5x - 12)$ $2x-3$ $x+4$

$= (x+1)(2x-3)(x+4)$ $\therefore x = -1, \frac{3}{2},$



QUESTION 4

$\alpha+\beta+\gamma = -\frac{b}{a} = \frac{6}{3} = 2$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} = \frac{1}{3}$

$\alpha\beta\gamma = -\frac{d}{a} = -\frac{2}{3}$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{\frac{1}{3}}{-2/3}$

$= -\frac{1}{2}$

.) i) $y = \frac{x^2}{4}$

$y' = \frac{2x}{4} = \frac{x}{2}$

at $x = 2t$, $y' = \frac{2t}{2} = t$

$y - y_1 = m(x - x_1)$

$y - t^2 = t(x - 2t)$

$y - t^2 = tx - 2t^2$

$0 = tx - y - t^2$

ii) If $y=0$, $tx - 0 - t^2 = 0$

$x = \frac{t^2}{t} = t$

$\therefore T(t, 0)$

midpoint $= \left(\frac{2t+t}{2}, \frac{t^2+0}{2}\right)$

$= \left(\frac{3t}{2}, \frac{t^2}{2}\right)$

ii) $x = \frac{3t}{2}$ $\quad y = \frac{t^2}{2}$

$t = \frac{2x}{3}$ $\quad = \left(\frac{2x}{3}\right)^2 \div 2$

$y = \frac{4x^2}{9} \times \frac{1}{2}$

$18y = 4x^2$

$9y = 2x^2$

c) i) $\angle ADC = 90°$ angle in a semi-circle

ii) $\angle DCA = 180 - 90 - 52$ (angle sum of Δ)

$= 38°$

$\angle DBA = 38°$ (angles in the same segment are equal)

QUESTION 5

a) 3% pa $= \frac{1}{4}$% pm $= 0.0025$

i) $A = P(1+r)^n$

$= 150(1.0025)^{72}$

$= \$179.54$

ii) $A_1 = 150(1.0025)^{72}$

$A_2 = 150(1.0025)^{71}$

$\vdots$

$A_{72} = 150(1.0025)^1$

Total $A = 150(1.0025 + 1.0025^2 + \ldots + 1.0025^{72})$

$= 150\left(\frac{1.0025(1.0025^{72} - 1)}{1.0025 - 1}\right)$

$= \$11846.45$

b) If $n=1$, LHS $= \frac{1}{2\times3} = \frac{1}{6}$

RHS $= \frac{1}{2(3)} = \frac{1}{6} =$ LHS

$\therefore$ true for $n=1$

Assume true for $n=k$

ie $\frac{1}{2\times3} + \frac{1}{3\times4} + \ldots + \frac{1}{(k+1)(k+2)} = \frac{k}{2(k+2)}$

To prove true for $n=k+1$

ie $\frac{1}{2\times3} + \ldots + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)} = \frac{k+1}{2(k+3)}$

LHS $= \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)}$

$= \frac{k(k+3) + 1(2)}{2(k+2)(k+3)}$

$= \frac{k^2+3k+2}{2(k+2)(k+3)}$

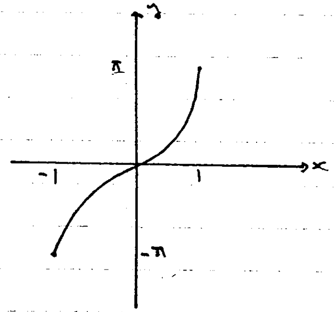
$= \frac{\cancel{(k+2)}(k+1)}{2\cancel{(k+2)}(k+3)}$

$= \frac{k+1}{2(k+3)}$

$\therefore$ true for $n = k+1$.

Since statement is true for $n=1$ it is true for $n = 1+1 = 2$, & $n = 2+1 = 3$ etc.

c) i)



ii) $f\left(\frac{1}{\sqrt{2}}\right) = 2\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= 2 \cdot \frac{\pi}{4}$

$= \frac{\pi}{2}$

iii) $f(x) = 2\sin^{-1} x$

$f'(x) = 2 \cdot \frac{1}{\sqrt{1-x^2}}$

$f'\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{1-\frac{1}{2}}}$

$= \frac{2}{\sqrt{\frac{1}{2}}}$

$= 2 \div \frac{1}{\sqrt{2}}$

$= 2\sqrt{2}$

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{2} = 2\sqrt{2}\left(x - \frac{1}{\sqrt{2}}\right)$

$y - \frac{\pi}{2} = 2\sqrt{2}x - 2$

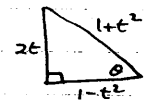
$y = 2\sqrt{2}x - 2 + \frac{\pi}{2}$

QUESTION 6

a) $\frac{d}{dx}\cos^{-1}(2x^3) = \frac{-1}{\sqrt{1-(2x^3)^2}} \cdot 6x^2$

$= \frac{-6x^2}{\sqrt{1-4x^6}}$

b) $\cos\theta(\tan\theta - \tan\frac{1}{2}\theta)$



$= \frac{1-t^2}{1+t^2}\left(\frac{2t}{1-t^2} - t\right)$

$= \frac{\cancel{1-t^2}}{1+t^2}\left(\frac{2t - t + t^3}{\cancel{1-t^2}}\right)$

$= \frac{t+t^3}{1+t^2}$

$= \frac{t(1+t^2)}{1+t^2}$

$= t = \tan\frac{1}{2}\theta$.

c) $\int_{\pi/4}^{\pi/2} \cos^2 x \, dx =$

$\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$\cos 2x + 1 = 2\cos^2 x$

$\frac{1}{2}\int_{\pi/4}^{\pi/2} (\cos 2x + 1)\, dx$

$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_{\pi/4}^{\pi/2}$

$= \frac{1}{2}\left[\left(\frac{1}{2}\sin\pi + \frac{\pi}{2}\right) - \left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4}\right)\right]$

$= \frac{1}{2}\left[\left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + \frac{\pi}{4}\right)\right]$

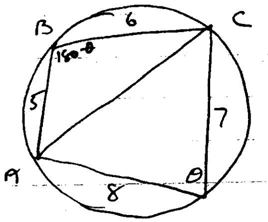
$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right]$

$= \frac{\pi}{8} - \frac{1}{4}$

3unit

) $\angle ABC = 180-\theta$ opposite $\angle$s of cyclic quadrilateral are supplementary.

)



$$\cos\theta = \frac{8^2+7^2-AC^2}{2.8.7}$$

$$= \frac{113-AC^2}{112}$$

$$AC^2 = 5^2+6^2-2.5.6.\cos(180-\theta)$$

$$= 61+60\cos\theta.$$

$$\therefore \cos\theta = \frac{113-(61+60\cos\theta)}{112}$$

$$112\cos\theta = 113-61-60\cos\theta$$

$$172\cos\theta = 52$$

$$\cos\theta = \frac{52}{172}$$

$$= \frac{13}{43}$$

QUESTION 7

a) $u = e^x$

$$\frac{du}{dx} = e^x$$

$$dx = \frac{du}{e^x}$$

$$\int \frac{e^x}{1+e^{2x}}dx = \int \frac{\cancel{u}}{1+u^2}\cdot\frac{du}{\cancel{u}}$$

$$= \int \frac{du}{1+u^2}$$

$$= \tan^{-1}u + c$$

$$= \tan^{-1}e^x + c$$

b) i) $R = \sqrt{a^2+b^2}$ $\quad a=\sqrt{3},\ b=1$

$$R = \sqrt{(\sqrt{3})^2+1^2}$$

$$= 2$$

$\tan\alpha = \frac{b}{a}$ $\quad 0<\alpha<\frac{\pi}{2}$.

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = 30^\circ = \frac{\pi}{6}.$$

$$R\sin(\theta-\alpha) = 2\sin\left(\theta-\frac{\pi}{6}\right)$$

ii) $2\sin\left(\theta-\frac{\pi}{6}\right) = 1$

$$\sin\left(\theta-\frac{\pi}{6}\right) = \frac{1}{2}$$

$\theta-\frac{\pi}{6} = \frac{\pi}{6}$, $\quad \theta-\frac{\pi}{6} = \frac{5\pi}{6}$

$\theta = \frac{\pi}{3}$ $\quad \theta = \pi$

c) i) $f(x) = \sqrt{x+6}$.

$$y = \sqrt{x+6}$$

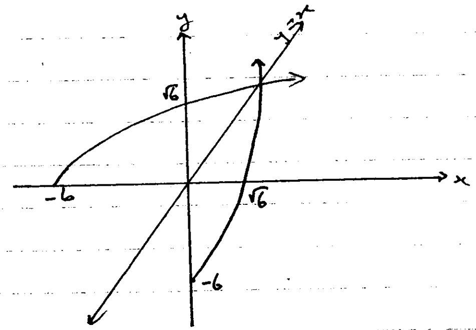
$$x = \sqrt{y+6}$$

$$x^2 = y+6$$

$$y = x^2-6$$

$$f^{-1}(x) = x^2-6$$

ii)



iii) D: $x \geq 0$

iv)

$y = x^2-6$ intersects with $y=x$.

$$x = x^2-6$$

$$x^2-x-6=0.$$

$y=\sqrt{x+6}$ intersects with $y=x$

$$x = \sqrt{x+6}$$

$$x^2 = x+6$$

$$x^2-x-6=0.$$

$$x^2-x-6=0$$

$$(x-3)(x+2)=0$$

$$x=3,-2$$

but $x\neq -2$

$\therefore x=3$

Point of intersection is $(3,3)$