Student Number: ....

SCEGGS Darlinghurst

2007
Higher School Certificate
Assessment Task 1

Mathematics-Extension I

Task Weighting: 25%

Outcomes Assessed: **PE2, PE3, HE2, HE7**

General Instructions

- Time allowed – 65 minutes
- **Start each question on a new page.**
- Attempt **all** questions and show all necessary working.
- Write your student number at the top of each page.
- Marks can be deducted for careless or badly arranged work
- Mathematical templates, geometrical equipment and scientific calculators may be used.

Question	Reasoning	Communication	Total
1	/4		/10
2	/3	/3	/12
3	/8	/4	/12
4	/7	/5	/12
Total	/22	/12	/46

Average:__________ St. Dev.:__________ Rank:__________

Parent's Signature__________________________

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Question 1 (10 Marks) **Marks**

a) Find the general solution, in terms of π, to: **2**

$$2\sin x = \sqrt{3}$$

b) Evaluate $\lim_{x\to 0} \dfrac{\sin 3x}{2x}$ **1**

c) i) Show that $(x+2)$ is a factor of the polynomial: **2**

$$P(x) = 4x^3 + x^2 - 11x + 6$$

and hence solve $P(x) = 0$ completely.

ii) Hence solve $4\cos^3\theta + \cos^2\theta = 11\cos\theta - 6$ for $0 \le \theta \le 2\pi$ **2**

Give answers to 2 decimal places.

d) i) How many eleven-letter arrangements can be made using the letters of the word: **1**

YARRAWARRAH

ii) What is the probability that an arrangement of the letters, chosen at random, has all the A's next to each other? **2**

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Question 2 (12 Marks) **Marks**

a) Given α, β, γ are the roots of the equation $3x^3 - 4x^2 + 7x - 5 = 0$.

Write down the values of:

i) $\alpha + \beta + \gamma$ **1**

ii) $\alpha\beta + \beta\gamma + \gamma\alpha$ **1**

iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

iv) $(\alpha - 1)(\beta - 1)(\gamma - 1)$ **2**

b) Find the roots of $4x^3 - 13x^2 - 13x + 4 = 0$ given that they are the first three terms of a geometric series. **3**

c) Prove by Mathematical Induction that: **3**

$$\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \frac{1}{9 \times 13} + \ldots\ldots\ldots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1} \quad \text{for } n \geq 1$$

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Question 3 (12 Marks) **Marks**

a) The polynomial $P(x) = x^3 - x^2 - 2x - 3$ has the same remainder when divided by $(x+a)$ and $(x-2a)$. **2**

Find the non-zero values of a.

b) i) Express $\tan 2A$ in terms of $\tan A$ **1**

ii) Hence, or otherwise, show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$ **2**

c) There are 18 students in a kindergarten class. Each week the teacher appoints three students to be a lunch monitor, class captain and recycling collector.

i) How many different appointments can the teacher make to the three positions in the first week? **2**

ii) Erin and Amy are both students in the class. What is the probability that both of them are appointed to any of the three positions at this time? **2**

d) Prove by mathematical induction that $2 \times 4^{2n+1} + 3^{3n+1}$ is divisible by 11 for all positive integers $n \geq 1$ **3**

START A NEW PAGE

Question 4 (12 Marks) **Marks**

a) A standard pack of 52 cards consists of 13 cards of each of the four suits: spades, hearts, clubs and spades.

i) In how many ways can seven cards be selected without replacement so that exactly 4 are diamonds and 3 are hearts? (Note: The order of the selection is not important) **2**

ii) In how many ways can seven cards be selected without replacement if at least five must be of the same suit? (Note: The order of the selection is not important) **2**

iii) Explain the calculations in your solution for part ii) **2**

b) i) Express $5\cos x - 3\sin x$ in the form $R\cos(x+a)$ **1**

ii) Hence find the minimum value of $5\cos x - 3\sin x$ and the value of x, in the domain $0 \le x \le \pi$, where this minimum occurs. (Give your answer to 2 decimal places) **3**

iii) Graph $y = 5\cos x - 3\sin x$ *for* $0 \le x \le \pi$. Label the x and y intercepts, the turning point and the end points. **2**

END OF TEST

2007 HSC Assessment Task 1 – Extension 1

Q1 a) $2\sin x = \sqrt{3}$

$\sin x = \frac{\sqrt{3}}{2}$ ✓

$x = n\pi + (-1)^n\left(\frac{\pi}{3}\right)$ ✓

b) $\lim_{x \to 0} \frac{\sin 3x}{2x}$

$= \frac{1}{2} \lim_{x \to 0} \frac{\sin 3x}{x}$

$= \frac{3}{2} \lim_{x \to 0} \frac{\sin 3x}{3x}$

$= \frac{3}{2} \times 1$ as $\lim_{x \to 0} \frac{\sin ax}{ax} = 1$

$= \frac{3}{2}$ ✓

c) i) $P(-2) = 4(-2)^3 + (-2)^2 - 11(-2) + 6$

$= -32 + 4 + 22 + 6$ ✓

$= 0$

$\therefore$ by the factor theorem

as $P(-2) = 0$ $(x+2)$ is a factor

$$
\begin{array}{r}
4x^2 - 7x + 3 \\
x+2 \,\big)\, \overline{4x^3 + x^2 - 11x + 6} \\
\underline{4x^3 + 8x^2} \\
-7x^2 - 11x + 6 \\
\underline{-7x^2 - 14x} \\
3x + 6 \\
\underline{3x + 6} \\
0
\end{array}
$$

$\therefore P(x) = (x+2)(4x^2 - 7x + 3)$

$= (x+2)(4x-3)(x-1)$

$\therefore P(x) = 0$

$(x+2)(4x-3)(x-1) = 0$

$x = -2 \quad x = \frac{3}{4} \quad x = 1$ ✓

c) ii) $4\cos^3\theta + \cos^2\theta = 11\cos\theta - 6$

let $x = \cos\theta$

$4x^3 + x^2 = 11x - 6$

$4x^3 + x^2 - 11x + 6 = 0$

$(x+2)(x-1)(4x-3) = 0$ from i)

$\therefore \cos\theta = -2 \qquad \cos\theta = 1 \qquad \cos\theta = \frac{3}{4}$ ✓

no soln $\qquad \theta = 0, 2\pi \qquad \theta = 0.72$

$\theta = 2\pi - 0.72$

$= 5.56$

$\therefore \theta = 0, 0.72, 5.56, 6.28.$ ✓ Reas. - 2

d) i) No of arrangements $= \frac{11!}{4!\,4!}$

$= 69300$ ✓

No of arrangements with AAAA $= 8 \times \frac{7!}{4!}$ ✓

$= 1680$

$P(AAAA) = \frac{1680}{69300}$

$= \frac{4}{165}$ ✓ Reason - 2

Q2 a) i) $\alpha + \beta + \gamma = \frac{-b}{a}$

$= -\frac{(-4)}{3}$

$= \frac{4}{3}$ ✓

ii) $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$

$= \frac{7}{3}$ ✓

iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma}{\alpha\beta\gamma} + \frac{\alpha\gamma}{\alpha\beta\gamma} + \frac{\alpha\beta}{\alpha\beta\gamma}$

$= \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$ ✓

$= \frac{7/3}{5/\,}$ $= \frac{7}{5}$ ✓

a) iv) $(\alpha-1)(\beta-1)(\gamma-1)$

$(\alpha\beta-\alpha-\beta+1)(\gamma-1)$

$\alpha\beta\gamma-\alpha\beta-\alpha\gamma-\beta\gamma+\gamma+\alpha+\beta-1$ ✓

$= \frac{5}{3} - \frac{7}{3} + \frac{4}{3} - 1$

$= -\frac{1}{3}$ ✓

b) let the roots be $\frac{a}{r}, a, ar$

$\frac{a}{r} \times a \times ar = -\frac{4}{4}$

$a^3 = -1$

$a = -1$ ✓

$\frac{a}{r} + a + ar = \frac{13}{4}$

$-\frac{1}{r} - 1 - r = \frac{13}{4}$

$-1 - r - r^2 = \frac{13r}{4}$

$-4 - 4r - 4r^2 = 13r$

$4r^2 + 17r + 4 = 0$ ✓

$(4r+1)(r+4) = 0$

$r = -\frac{1}{4}$ $\quad r = -4$

$\therefore$ roots are $4, -1, \frac{1}{4}$ ✓ Reas - 3

c) Step 1: Show result is true for $n=1$

LHS $= \frac{1}{1\times 5} = \frac{1}{5}$ $\quad$ RHS $= \frac{1}{4\times1+1}$

$= \frac{1}{5}$

$\therefore$ LHS = RHS ✓

$\therefore$ the result is true for $n=1$

Step 2: Assume the result is true for $n=k$

i.e. $\frac{1}{1\times5} + \frac{1}{5\times9} + \dots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$

Step 3: Show the result is true for $n=k+1$

i.e. $\frac{1}{1\times5}+\frac{1}{5\times9}+\dots+\frac{1}{(4k-3)(4k+1)}+\frac{1}{(4k+1)(4k+5)}=\frac{k+1}{4k+5}$

$$LHS = \frac{1}{1\times5}+\frac{1}{5\times9}+\dots+\frac{1}{(4k-3)(4k+1)}+\frac{1}{(4k+1)(4k+5)}$$

$$= \frac{k}{4k+1}+\frac{1}{(4k+1)(4k+5)} \quad \text{(from step 2)} \checkmark$$

$$= \frac{k(4k+5)+1}{(4k+1)(4k+5)}$$

$$= \frac{4k^2+5k+1}{(4k+1)(4k+5)}$$

$$= \frac{\cancel{(4k+1)}(k+1)}{\cancel{(4k+1)}(4k+5)} \quad \checkmark$$

$$= \frac{k+1}{4k+5}$$

Comm. -3

$= RHS$

$\therefore$ the result is true for $n=k+1$

$\therefore$ by the Principle of Induction the result is true for $n \geq 1$

Q3 a) $P(-a) = (-a)^3-(-a)^2-2(-a)-3$

$= -a^3-a^2+2a-3$

$P(2a) = (2a)^3-(2a)^2-2(2a)-3$

$= 8a^3-4a^2-4a-3$

$\therefore$ if $P(-a)=P(2a)$

then $-a^3-a^2+2a-3 = 8a^3-4a^2-4a-3$ ✓

$0 = 9a^3-3a^2-6a$

$0 = 3a(3a^2-a-2)$

$0 = 3a(3a+2)(a-1)$

$\therefore$ non-zero values of a are

$a=1$ and $a=-\frac{2}{3}$ ✓ Reas - 2

b) i) $\tan(A+A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A}$

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ ✓ Comm.-1

ii) let $A = \frac{\pi}{8}$

$\therefore \tan\frac{\pi}{4} = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

$1 = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ ✓

$1-\tan^2\frac{\pi}{8} = 2\tan\frac{\pi}{8}$

$\tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$

let $t = \tan\frac{\pi}{8}$

$t^2 + 2t - 1 = 0$

$t = \dfrac{-2 \pm \sqrt{2^2 - 4\times1\times-1}}{2\times1}$

$= \dfrac{-2\pm\sqrt{8}}{2}$

$= \dfrac{-2\pm2\sqrt{2}}{2}$

$= -1\pm\sqrt{2}$

since $\frac{\pi}{8}$ is in the first quadrant

$\tan\frac{\pi}{8} > 0$

$\therefore \tan\frac{\pi}{8} = \sqrt{2}-1$ ✓ Reas-2

c) i) No. of appointments $= {}^{18}C_3 \times 3!$ ✓

$= 4896$ ✓ Reas-2

ii) No. of arrangements with Ern & Amy

$= {}^{16}C_1 \times 1 \times 1 \times 3!$

$= 96$ ✓

$\therefore P(\text{Ern and Amy}) = \dfrac{96}{4896}$

$= \dfrac{1}{51}$ ✓ Reas-2

d) Step 1: show true for $n=1$

for $n=1$

$$2\times 4^{3} + 3^{4}$$
$$= 2\times 64 + 81$$
$$= 128 + 81$$
$$= 209 \quad \checkmark$$
$$= 11\times 19$$

$\therefore$ the result is true for $n=1$

Step 2: assume the result is true for $n=k$

i.e. $2\times 4^{2k+1} + 3^{3k+1} = 11M$

where M is any integer

Step 3: show the result is true for $n=k+1$

i.e. $2\times 4^{2k+3} + 3^{3k+4} = 11Q$

where Q is any integer

$$\text{LHS} = 2\times 4^{2k+3} + 3^{3k+4}$$
$$= 4^2\left(2\times 4^{2k+1}\right) + 3^3\left(3^{3k+1}\right)$$
$$= 16\left(11M - 3^{3k+1}\right) + 27\left(3^{3k+1}\right) \quad (\text{from step } 2) \quad \checkmark$$
$$= 176M - 16\times 3^{3k+1} + 27\times 3^{3k+1}$$
$$= 176M + 11\times 3^{3k+1}$$
$$= 11\left(16M + 3^{3k+1}\right) \quad \checkmark$$
$$= 11Q$$ where Q is any integer

as M and k are integers

$\therefore$ by the Principle of Induction the result is true for $n\geq 1$

Comm - 3

Q4 a) i) No. of ways = ${}^{13}C_4 \times {}^{13}C_3$ ✓

$= 204490$ ✓ Reas − 2

ii) 5 the same: No. of ways = ${}^{13}C_5 \times {}^{39}C_2$

$= 953\,667$

6 the same: No of ways = ${}^{13}C_6 \times {}^{39}C_1$

$= 66924$ ✓

7 the same: No of ways = ${}^{13}C_7$

$= 1716$

$\therefore$ total $= 1\,022\,307$

$\therefore$ total no of ways $= 1022307 \times 4$ ✓

$= 4\,089\,228$ Reas − 2

iii) • "at least 5 the same" means 5, 6, or 7 the same

• from 1 suits of 13 choose either 5, 6 or 7 and the remainders (2, 1 or 0) from 39 cards

• had to multiply by 4 for the 4 different suits. any 2 of these ✓✓

Comm − 2

b) i) $5\cos x - 3\sin x = R\cos x\cos a - R\sin x\sin a$

$\therefore R\cos a = 5 \ldots ①$

$R\sin a = 3 \ldots ②$

② ÷ ① $\tan a = \frac{3}{5}$

$a = 0.54$

$R^2\cos^2 a + R^2\sin^2 a = 5^2 + 3^2$

$R^2(\cos^2 a + \sin^2 a) = 25 + 9$

$R^2 = 34$ ✓ Comm − 1

$R = \sqrt{34}$ $\therefore$ $5\cos x - 3\sin x = \sqrt{34}\cos(x + 0.54)$

ii) min value $= -\sqrt{34}$ ✓

$\therefore \sqrt{34}\cos(x + 0.54) = -\sqrt{34}$

$\cos(x + 0.54) = -1$ ✓

$x + 0.54 = \pi$ Reas − 3

$x = 2.60$ (to 2 d.p.). ✓

iii) Sketch $y = 5\cos x - 3\sin x$

$\therefore$ sketch $y = \sqrt{34}\cos(x + 0.54)$

turning point = min ~~turning~~ point

$\therefore (2.60, -\sqrt{34})$

y-int : $x = 0$

$y = \sqrt{34}\cos(0 + 0.54)$

$= 5.$

x-int : $y = 0$

$\sqrt{34}\cos(x + 0.54) = 0$

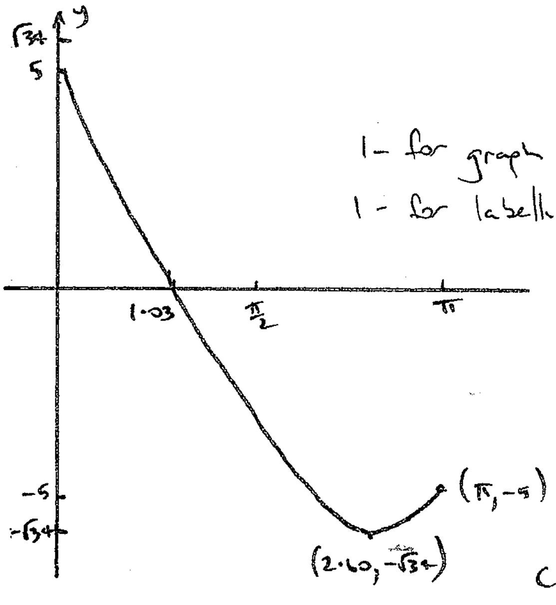
$x + 0.54 = \frac{\pi}{2}$

$x = 1.03$

endpoint $x = \pi$

$y = \sqrt{34}\cos(\pi + 0.54)$

$= -5$



1 – for graph

1 – for labelling

Comm – 2