Name: ____________________

SCEGGS Darlinghurst

Term 1, 2008
Tuesday 1st April

EXTENSION 1 MATHEMATICS

Higher School Certificate
Assessment Task 1

Task Weighting : 30 %

Outcomes Assessed: PE2, PE3, PE4, HE2, HE6, HE7

General Instructions

- Time allowed - 70 minutes
- Write your name at the top of each page
- Start each question on a new page
- Attempt **all** questions.
- Marks may be deducted for careless or badly arranged work
- Approved calculators should be used
- Mathematical templates and geometrical equipment may be used.

Question 1				/11
Question 2				/10
Question 3				/11
Question 4				/12
TOTAL				/44

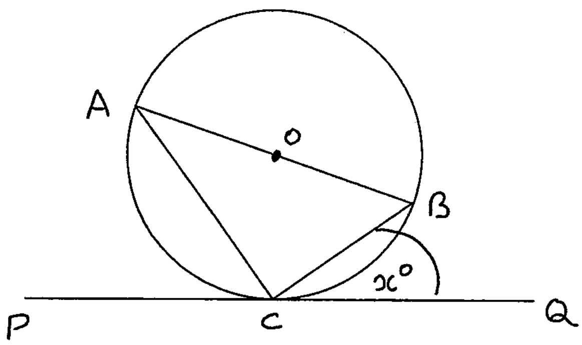
Question 1 (11 Marks) Marks

(a) Use the substitution $u = x^2 + 1$ to find 3

$$\int x(x^2+1)^5\,dx$$

(b) Find the equation of the chord of the parabola joining the points with 2
parameters $t = \frac{1}{2}$ and $t = -1$ on $x = 8t, y = 4t^2$

(c) A, B and C lie on a circle with centre, O.
PQ is a tangent to the circle at C.
If $\angle ABC = 38°$, find the value of x giving reasons. (Copy the diagram) 3



(Not to scale)

(d) (i) A committee is to be formed consisting of 4 Year 11 girls and 6 Year 12 girls. In how many ways can the committee be chosen from 30 Year 11 and 40 Year 12 girls? 1

(ii) A sub-committee of 5 girls is then chosen from the committee of 10. How many possible sub-committees are there in which Year 12 have the majority? 2

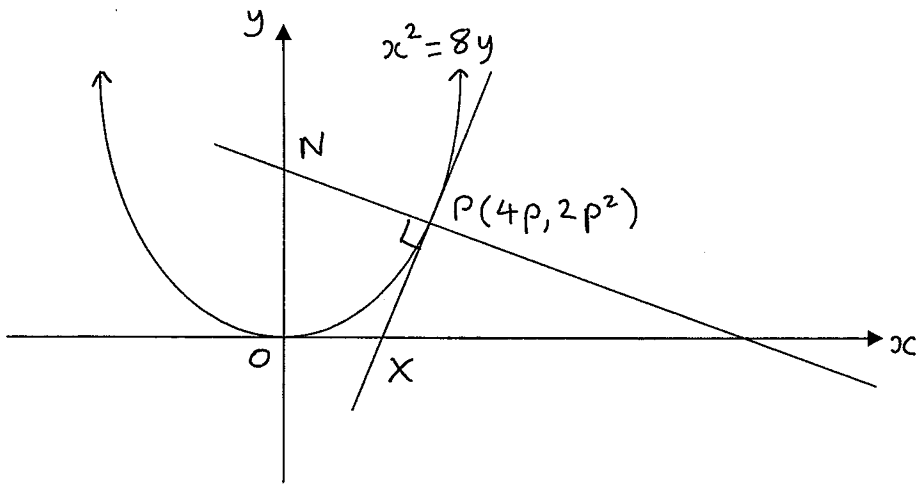
Question 2 (10 marks) START A NEW PAGE | Marks

(a) (i) How many nine-letter arrangements can be made using the letters of the word SCHOOLIES ? | 1

(ii) In how many of these arrangements do the vowels appear together? | 1

(iii) In how many arrangements does the word COOL appear? | 1

(b) The diagram shows the graph of the parabola $x^2 = 8y$. The tangent to the parabola at $P(4p, 2p^2)$ cuts the x-axis at X. The normal to the parabola at P cuts the y-axis at N.



Copy the diagram onto your answer page.

(i) Find the equation of the tangent at P. | 2

(ii) The normal at P has the equation $x + py = 4p + 2p^3$
Find the coordinates of N. | 1

(iii) Let M be the midpoint of NX.
Find the coordinates of M. | 2

(iv) Show that the locus of M is a parabola.
.Find its vertex and focal length. | 2

Question 3 (11 Marks) START A NEW PAGE — Marks

(a) Eight people are to be seated at a round table.

(i) How many seating arrangements are possible if there are no restrictions? **1**

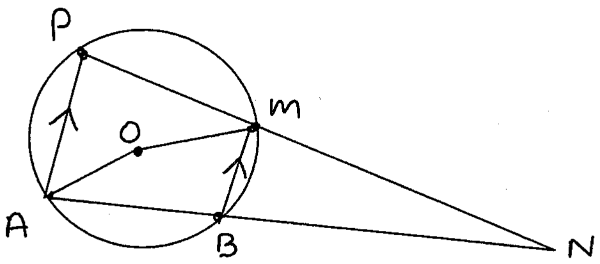
(ii) Two people, Ken and Barbie, refuse to sit next to each other. How many seating arrangements are then possible? **1**

(iii) If the eight people consist of four sets of twins, how many seating arrangements are possible if each set of twins must sit opposite each other? **1**

(b) Use Mathematical Induction to show that for all positive integers, $n \geq 1$. **3**

$$\frac{3}{1\times 2\times 2}+\frac{4}{2\times 3\times 2^2}+\ldots+\frac{n+2}{n\times(n+1)\times 2^n} = 1-\frac{1}{(n+1)\times 2^n}$$

(c) In the diagram below, O is the centre of the circle and $AP||BM$. AB and PM meet at N.



Copy the diagram onto your answer page.

(i) Prove ΔNAP is isosceles **2**

(ii) Prove $MOAN$ is a cyclic quadrilateral. **3**

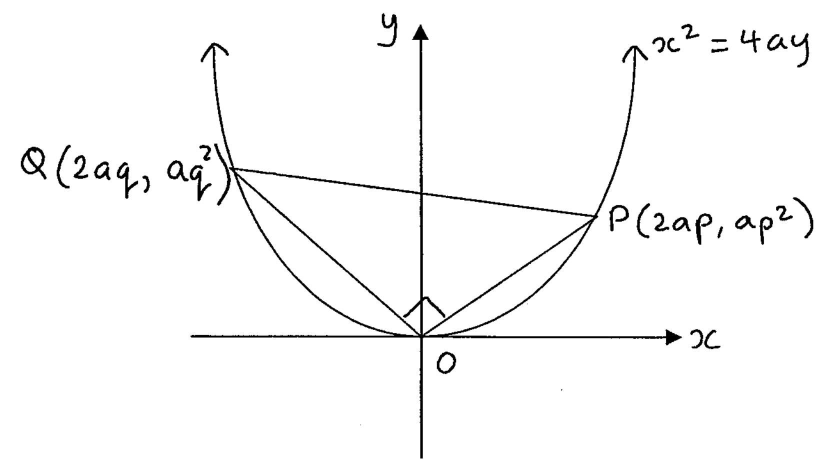
Question 4 (12 marks) START A NEW PAGE | Marks

(a) Use the substitution $u = x + 1$ to evaluate

$$\int_0^3 \frac{x-2}{\sqrt{x+1}}\,dx$$ 3

(b) Use Mathematical Induction to prove that $4 \times 2^n + 3^{3n}$ is divisible by 5 for all integers n, $n \geq 0$. 3

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The chord PQ subtends a right angle at the vertex.



Copy the diagram into your answer booklet.

(i) Show that $pq = -4$ 2

(ii) The normals at P and Q meet at N.
Show that N has coordinates $(-apq(p+q),\ a(p^2 + pq + q^2 + 2))$ 2

You may assume these equations. Do not find them.
Normal at P: $x + py = 2ap + ap^3$
Normal at Q: $x + qy = 2aq + aq^3$

(iii) Hence, determine the locus of N as P and Q move along the parabola. Give an accurate description of the locus of N. 2

END OF PAPER

Year 12 Ext. 1 Term 1 Task #1 2008

Question 1.

Calc /3
Comm /3
Reas /2

a) $u = x^2 + 1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$$\int x(x^2+1)^5\,dx$$
$$= \tfrac{1}{2}\int (x^2+1)^5 \cdot 2x\,dx$$
$$= \tfrac{1}{2}\int u^5 \cdot du$$
$$= \tfrac{1}{2} \times \frac{u^6}{6} + C$$
$$= \frac{u^6}{12} + C$$
$$= \frac{(x^2+1)^6}{12} + C$$

Well done by most students, unless you really messed it up.

It was surprising that some people made mistakes integrating here.

Don't forget the final answer must be written in terms of x.

(Calc 3)

b) parabola $x = 8t,\ y = 4t^2$

Find the coordinates of endpoints

when $t = \frac{1}{2}$ $\quad (8 \times \frac{1}{2},\ 4 \times (\frac{1}{2})^2) = (4, 1)$

when $t = -1$ $\quad (8 \times -1,\ 4 \times (-1)^2) = (-8, 4)$

Equation of chord $(4,1)\ (-8,4)$

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$
$$\frac{y-1}{x-4} = \frac{4-1}{-8-4}$$
$$\frac{y-1}{x-4} = \frac{3}{-12}$$
$$\frac{y-1}{x-4} = -\frac{1}{4}$$
$$4y - 4 = -x + 4$$
$$x + 4y - 8 = 0$$

Some people found these points but couldn't find the equation of the straight line.

An easy question which could appear early in a HSC paper.

c) $\angle ACB = 90°$ (angle in a semicircle $= 90°$)

$\angle BAC = 180° - (38 + 90)°$ ($\angle$ sum $\Delta = 180°$)

$= 52°$

$\angle BCQ = \angle BAC$ ($\angle$ between a tangent and a chord equals the angle in the alternate segment.)

$= 52°$

Well done! An easy question.

(Comm 3)

d) i) $^{30}C_4 \times {}^{40}C_6 = 1.052 \times 10^{11}$

choose Yr 11 — choose Yr 12

Very well done.

ii) Committee 4 Year 11, 6 Year 12

Year 12 majority in subcommittee of 5

Yr 12	Yr 11
5	0
4	1
3	2

$^6C_5 \times {}^4C_0 = 6$

$^6C_4 \times {}^4C_1 = 60$

$^6C_3 \times {}^4C_2 = 120$

TOTAL $= 186$

✓ one case

✓ correct solution.

Read the question carefully. A subcommittee is being formed from the group of 10.

(Reas 2)

Question 2.

Reas /3
Calc /1

a) i) SCHOOLIES

9 letters
2S
2O

No. arrangements $= \frac{9!}{2!\,2!}$

$= 90720$ ✓

Just take care when counting double letters!

ii) $\boxed{V\ V\ V\ V}$ _ _ _ _ _

Treat as 6 groups

4 vowels
5 consonants.
2S
2O

No. arrangements with vowels together $= \frac{6! \times \overset{\text{arrange vowels}}{4!}}{\underset{\text{repeated S}}{2!}\ \underset{\text{O}}{2!}}$

$= 4320$ ✓

In this question you must multiply by arrangements of the vowels OOIE — and there are two Os!!!

iii) $\boxed{C\ O\ O\ L}$ _ _ _ _ _

Fix the letters in the word COOL

Treat as six groups, 2S

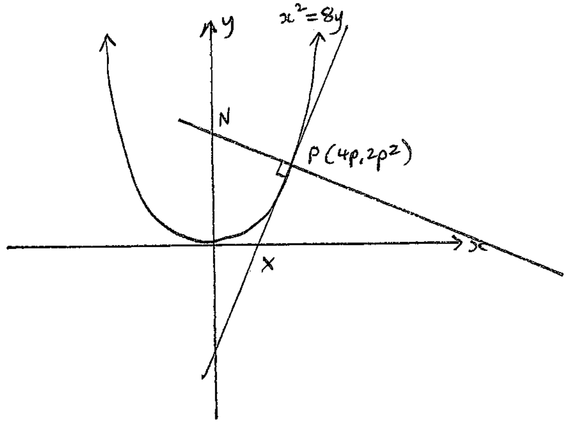
No. arrangements $= \frac{6!}{\underset{\text{repeated S}}{2!}}$

$= 360$ ✓

Reas /3

In this question you don't multiply for rearrangements of the letters COOL.

b)



All of part b was done well by most.

- the algebra in this question was easy & should be done quickly + efficiently.

i) $x^2 = 8y$

$y = \frac{x^2}{8}$

$\frac{dy}{dx} = \frac{2x}{8}$

$= \frac{x}{4}$

At $P(4p, 2p^2)$

gradient tangent

$m_T = \frac{4p}{4}$

$= p$ ✓

Calc /1

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - 2p^2 = p(x - 4p)$

$y - 2p^2 = px - 4p^2$

$y = px - 2p^2$ ✓

ii) Normal cuts y axis at N.

substitute $x = 0$ into $x + py = 4p + 2p^3$

$0 + py = 4p + 2p^3$

$y = \frac{4p + 2p^3}{p}$

$= 4 + 2p^2$

$\therefore N(0, 4 + 2p^2)$ ✓

iii) Tangent cuts x axis at X when $y = 0$

$0 = px - 2p^2$

$px = 2p^2$

$x = 2p.$

$\therefore X(2p, 0)$ ✓

$N(0,\ 4+2p^2)$ $\quad X(2p,0)$

midpoint

$$M=\left(\frac{0+2p}{2},\ \frac{4+2p^2+0}{2}\right)$$

$$=(p,\ 2+p^2)$$

iv) Locus of M

parametric form $\begin{cases} x=p & ① \\ y=2+p^2 & ② \end{cases}$

Eliminate p for cartesian form.

sub. ① into ②

$$y=2+x^2$$

$\therefore$ Locus is a parabola.

$$x^2=y-2$$

$$x^2=4\cdot\tfrac{1}{4}(y-2)$$

$\left.\begin{array}{l}\text{Vertex } (0,2)\\ \text{Focal length } a=\tfrac{1}{4}\end{array}\right\}$

Needed both the vertex & the focal length for 1 mark!

Question 3.

Reas /6
Comm /5

a) i) No restrictions $=\dfrac{8!}{8}$

$$=7!$$

$$=5040$$

ii) 

$$\frac{\underset{\text{Ken}}{8}\times\underset{\text{Barbie}}{5}\times\underset{\text{seat the rest}}{6!}}{\underset{\text{round table}}{8}}$$

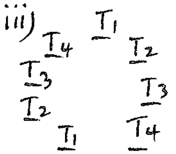
Fix Ken

$$=1\times5\times6!$$

$$=3600$$

\# arrangements = Total − Ken + Barbie sitting together

was another method done successfully by many.

iii) 

$$\frac{\underset{\text{first }T_1}{8}\times\underset{\text{first }T_2}{6}\times\underset{\text{first }T_3}{4}\times\underset{\text{first }T_4}{2}}{\underset{\text{circle}}{8}}=48$$

OR

$$\underset{\text{fix first twin couple}}{1}\times\underset{\text{first other twins}}{3!}\times\underset{\text{second twin}}{1}\times\underset{\text{couple swap}}{(2!)^3}$$

Many had too many or too few factors of 2 in their answer which was unfortunate.

Reas /3

b) Prove that

$$\frac{3}{1\times2\times2}+\frac{4}{2\times3\times2^2}+\ldots+\frac{n+2}{n(n+1)2^n}=1-\frac{1}{(n+1)2^n}$$

Prove true for $n=1$

$$LHS=\frac{3}{1\times2\times2}=\frac{3}{4}$$

$$RHS=1-\frac{1}{2\times2}=1-\frac{1}{4}=\frac{3}{4}$$

$LHS=RHS$

$\therefore$ True for $n=1$

Assume true for $n=k$

$$\frac{3}{1\times2\times2}+\frac{4}{2\times3\times2^2}+\ldots+\frac{k+2}{k(k+1)2^k}=1-\frac{1}{(k+1)2^k}$$

Prove true for $n=k+1$

ie

$$\frac{3}{1\times2\times2}+\frac{4}{2\times3\times2^2}+\ldots+\frac{(k+1)+2}{(k+1)(k+2)2^{k+1}}=1-\frac{1}{(k+2)2^{k+1}}$$

$$LHS=\underbrace{\frac{3}{1\times2\times2}+\frac{4}{2\times3\times2^2}+\ldots+\frac{k+2}{k(k+1)2^k}}_{\text{by the assumption}}+\frac{k+3}{(k+1)(k+2)2^{k+1}}$$

$$=1-\frac{1}{(k+1).2^k}+\frac{k+3}{(k+1).(k+2).2.2^k}$$

common denominator

$$=1-\left(\frac{1}{(k+1)\cdot2^k}-\frac{k+3}{(k+1)(k+2)\cdot2\cdot2^k}\right)$$

$$=1-\left(\frac{2(k+2)-(k+3)}{(k+1)(k+2)2\cdot2^k}\right) \quad ✓$$

$$=1-\left(\frac{k+1}{(k+1)(k+2)2^{k+1}}\right)$$

$$=1-\frac{1}{(k+2)2^{k+1}} \quad ✓$$

$= RHS$

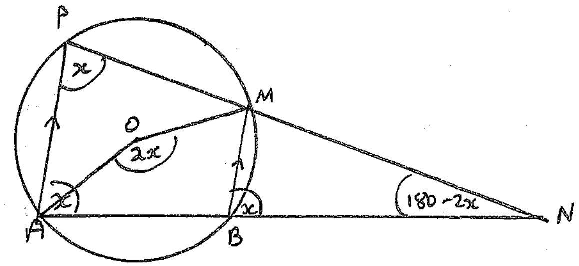
$\therefore$ If the statement is true for $n=k$, it is true for $n=k+1$. Since it is true for $n=1$, it is true for $n=2$ and so on.
Hence by the principle of mathematical induction, it is true for all $n\geq1$

Reas /3

Two marks were awarded for this proof. For the first mark you needed to make a common denominator, it wasn't enough to simply substitute in the assumption.

Not everyone got the second mark because of poor algebra with negatives / fractions. This is an easy induction as long as you are fluent working with negatives! These are common questions so practise them!

c)



i) Let $\angle MBN = x$

$\angle PAN = \angle MBN = x$ ✓
(Corresponding $\angle$s are equal, since // lines PA // MB)

(*)
$\angle PMB = 180 - x$ (opposite $\angle$s in a cyclic quadrilateral are supplementary)
$\angle APM = 180° - (\angle PMB)$
$\angle APM = x$ (cointerior $\angle$s are supplementary since // lines. AP // BM) ✓

OR $\angle APM = \angle MBN$
$= x$
(The interior angle in a cyclic quadrilateral equals the opposite exterior angle.)

$\therefore$ ΔNAP is isosceles (2 angles equal).

ii) $\angle MOA = 2\angle APM$
$= 2x$
($\angle$ at centre is double the angle at the circumference.) ✓

In ΔAPN, $\angle ANP = 180° - (x+x)$
$= 180° - 2x$
($\angle$ sum $\Delta = 180°$) ✓

$\therefore$ MOAN is a cyclic quadrilateral.
Since opposite angles are supplementary ✓

Comm /5

An HSC question will always ask you to copy the diagram – not so you waste your time copying the diagram but so there is somewhere for you to label angles, show what you're doing and in the case you forget to tell the marker what you are calling angle x – they can still follow your argument!

SO COPY THE DIAGRAM & WRITE ON IT!

Overall this was a relatively easy circle geometry problem done well by most ... but nevertheless there was certainly room for improvement in terms of writing clear + logical arguments.

Question 4.

Calc /3
Comm /4
Reas /1

a) $\begin{cases} u = x+1 \rightarrow x = u-1 \\ \frac{du}{dx} = 1 \\ du = dx \end{cases}$

change limits

$x = 0 \quad u = 1$
$x = 3 \quad u = 4$

$$\int_0^3 \frac{x-2}{\sqrt{x+1}}\,dx$$

$$= \int_1^4 \frac{(u-1-2)}{\sqrt{u}}\,du$$

$$= \int_1^4 \frac{u-3}{u^{1/2}}\,du$$

$$= \int_1^4 \left(u^{\frac{1}{2}} - 3u^{-\frac{1}{2}}\right) du \quad ✓$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}} - \frac{3u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$

$$= \left[\frac{2}{3}u^{3/2} - 6u^{\frac{1}{2}}\right]_1^4 \quad ✓$$

$$= \left(\frac{2}{3}\times 4^{\frac{3}{2}} - 6\times 4^{\frac{1}{2}}\right) - \left(\frac{2}{3}\times 1^{3/2} - 6\times 1^{1/2}\right)$$

$$= \left(\frac{2}{3}\times 8 - 6\times 2\right) - \left(\frac{2}{3} - 6\right)$$

$$= \left(-6\frac{2}{3}\right) - \left(-5\frac{1}{3}\right) \quad ✓$$

$$= -1\frac{1}{3}$$

Reasonably well done. Just some silly mistakes but most people knew the method to use.

Calc 3

b) Prove that $4\times 2^n + 3^{3n}$ is divisible by 5 for all $n \geq 0$.

Prove true for $n=0$

$4\times 2^0 + 3^0 = 4\times 1 + 1$
$= 5$
which is divisible by 5

$\therefore$ true for $n=0$ ✓

Did you notice this proof starts with $n=0$? Highlight features like this in a question.

Assume true for $n=k$

$4\times 2^k + 3^{3k} = 5P$ (where P is some integer > 0)

rearrange ie. $4.2^k = 5P - 3^{3k}$

Prove true for $n=k+1$

ie. $4\times 2^{k+1} + 3^{3(k+1)} = 5Q$ (where Q is some integer > 0)

$LHS = 4\times 2^{k+1} + 3^{3(k+1)}$

$= 4.2^k.2 + 3^{3k+3}$

using assumption

$= (5P - 3^{3k}).2 + 3^{3k}.3^3$ ✓

$= 10P - 2.3^{3k} + 27.3^{3k}$

$= 10P + 25.3^{3k}$

$= 5(2P + 5.3^{3k})$

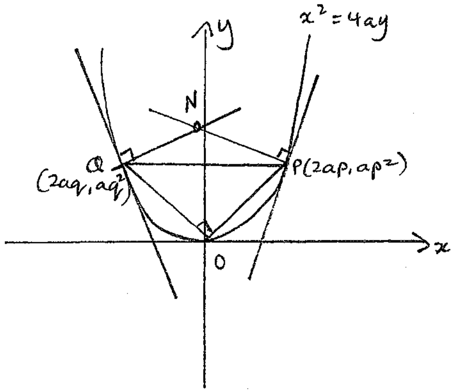
$= 5Q$ where Q is integer, $Q > 0$. (since P and k are integers) ✓

Rearrange the assumption and substitute it.

If true for $n=k$, then it is true for $n=k+1$.
Since it is true for $n=0$, it is then true for $n=1$ and so on.
Hence by the principle of mathematical induction, it is true for all $n \geq 0$.

Comm 3

c)



Some students confused this method with gradients of tangents.

$\left.\begin{matrix} m_{OQ} \\ m_{OP} \end{matrix}\right\}$ are not tangent gradients.

i) Using gradients $OQ \perp OP$

$$m_{OQ} \times m_{OP} = -1$$

$$\frac{(aq^2-0)}{(2aq-0)} \times \frac{(ap^2-0)}{(2ap-0)} = -1$$

$$\frac{a^2p^2q^2}{4a^2pq} = -1$$

$$\frac{pq}{4} = -1$$

$$\therefore \quad pq = -4$$

ii) Normal at P: $x + py = 2ap + ap^3$ ①

Normal at Q: $x + qy = 2aq + aq^3$ ②

Don't waste your time finding equations that are given!!!!!

Solve simultaneously ① - ②

$$py - qy = 2ap - 2aq + ap^3 - aq^3$$

$$y(p-q) = 2a(p-q) + a(p^3-q^3)$$

$$y = \frac{2a\cancel{(p-q)} + a\cancel{(p-q)}(p^2+pq+q^2)}{\cancel{(p-q)}}$$

$$y = 2a + a(p^2+pq+q^2)$$

$$= a(p^2+pq+q^2+2)$$

Substitute into ① to find x

$$x + ap(p^2+pq+q^2+2) = 2ap + ap^3$$

$$x + \cancel{ap^3} + ap^2q + apq^2 + \cancel{2ap} = \cancel{2ap} + \cancel{ap^3}$$

$$x = -ap^2q - apq^2$$

$$= -apq(p+q)$$

Careful algebra work here will lead to success. This is a standard question. Please practise solving like this.

$$\therefore N\left(-apq(p+q),\ a(p^2+pq+q^2+2)\right)$$

iii) Parametric form of N

using $pq=-4$ from part (i).

$$x = -apq(p+q)$$

$$= -a.-4(p+q)$$

$$= 4a(p+q)$$

$$y = a(p^2+pq+q^2+2)$$

$$= a(p^2-4+q^2+2)$$

$$= a(p^2+q^2-2)$$

cont'

$$\begin{cases} x = 4a(p+q) \\ y = a(p^2+q^2-2) \end{cases}$$

using $p^2+q^2 = (p+q)^2 - 2pq$

You should know how to use this substitution in locus questions.

this is a standard technique.

$$y = a((p+q)^2 - 2pq - 2)$$

Substitute $p+q = \frac{x}{4a}$ and $pq = -4$

Reas 1

$$y = a\left(\left(\frac{x}{4a}\right)^2 - 2\times-4 - 2\right)$$

$$y = a\left(\frac{x^2}{16a^2} + 8 - 2\right)$$

$$y = a\left(\frac{x^2}{16a^2} + 6\right)$$

$$y = \frac{x^2}{16a} + 6a$$

The locus is a parabola.

$$16ay = x^2 + 96a^2$$

$$x^2 = 16ay - 96a^2$$

$$x^2 = 16a(y - 6a)$$

$$x^2 = 4 \cdot 4a(y-6a)$$

Not many students got this far.

Congratulations if you made it :)

The locus is a parabola with vertex $(0, 6a)$

focal length $= 4a$

Comm 1