THE SCOTS COLLEGE

YEAR 12 EXTENSION 1 MATHEMATICS

PRE-TRIAL

APRIL 2006

WEIGHTING: 30%

TIME ALLOWED: **1½ HOURS (PLUS 5 MINUTES READING TIME)**

INSTRUCTIONS TO STUDENTS:

- START EACH QUESTION IN A NEW BOOKLET.
- BOARD APPROVED CALCULATORS MAY BE USED.
- MARKS MAY BE DEDUCTED FOR CARELESS WORKING.
- ALL NECESSARY WORKING MUST BE SHOWN.

Question 1 **(10 marks)**

a) Evaluate $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ **2**

b) Show that if p and q are the roots of the quadratic equation **2**
$2x^2+6x+3=m(2x+1)$, then $p+q+2pq=0$

c) $\int xe^{x^2}\,dx$ **1**

d) If $\log_8 2=\log_x 5$ find the value of x **2**

e) Solve the equation $\cos 2A=\cos A$ where $0\le A\le 2\pi$ **3**

Question 2 **(12 marks)**

a) i) Find a point where the curve $y=x^3-4x^2+8x-8$ crosses the x axis. **3**

ii) Show that this is the only point where the curve crosses the x axis.

b)

i. Sketch the graph of the function $y=\sin^{-1}\left(\frac{x}{2}\right)$ **1**

ii. State the domain and the range of the function **2**

iii. Find the exact equation of the tangent to the curve $y=\sin^{-1}\left(\frac{x}{2}\right)$ at the point where $x=1$ **3**

c) The point $P\left(t+3,t^2-5\right)$ lies on a parabola

i. Find the Cartesian equation of the parabola **2**

ii. Find the vertex of the parabola **1**

Question 3 (16 marks)

a)

i. Show that $\cos\sigma - \sqrt{3}\sin\sigma = 2\cos\left(\sigma + \frac{\pi}{3}\right)$ **2**

ii. Hence solve the equation $\cos\sigma - \sqrt{3}\sin\sigma = 1$ for σ in the interval $0 \le \sigma \le 2\pi$ **2**

b) Use the substitution $u = \sqrt{x}$ to evaluate $\int_1^3 \frac{dx}{(1+x)\sqrt{x}}$ **3**

Give your answer in exact form.

c)

i. Differentiate $x\cos^{-1}x - \sqrt{1-x^2}$ with respect to *x*. **3**

ii. Hence, evaluate $\int_0^1 \cos^{-1}x\, dx$ **2**

d) *ABCD* is a cyclic quadrilateral in which the opposite sides *AB* and *DC* are equal

i. Draw a diagram **1**

ii. Prove that the diagonals *AC* and *BD* are equal **3**

-------- Question four and five continued on back page --------

Question 4 (11 marks)

a) Find the sum of the first ten terms of the following Geometric series leaving your answer as a fraction. **2**

$4 - 2 + 1 - \frac{1}{2} + \ \ldots..$

b) Prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression if $\frac{a}{c} = \frac{a-b}{b-c}$ **3**

c) Find $\int \cos^2 x \quad dx$ **2**

d) The area enclosed between the curve $y = \sqrt{\frac{4}{1+x^2}}$ and the line $y = \sqrt{2}$ is rotated about the x axis. Find the volume of the solid of revolution generated. Give your answer in terms of π **4**

Question 5 (13 marks)

a) Use the method of mathematical induction to show that **4**

$1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots. + n^2 = \frac{1}{6}n(n+1)(2n+1)$

b)

i. Show that the equation of the tangent at the point $(2ap, ap^2)$ on the parabola $x^2 = 4ay$ is $y = px - ap^2$ **2**

ii. This tangent cuts the x axis at Q. S is the focus. Prove that $\angle PQS$ is a right angle. **3**

iii. Show that the centre of the circle passing through P, Q and S is the midpoint of SP **2**

iv. Show that the locus of the centre of the circle through P, Q and S is $2ay = a^2 + x^2$ **2**

END OF PAPER

Question 1

a) $\cos^{-1}\left(-\frac{1}{2}\right)$

$= \pi - \cos^{-1}\left(\frac{1}{2}\right)$ 1

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$ 1

b) $2x^2+6x+3 = 2mx+\cancel{2}m$

$2x^2+6x-2mx+3-m=0$

$2x^2+x(6-2m)+3-m=0$

$p+q = \frac{-B}{a}$ $\qquad$ $pq = \frac{c}{a}$

$p+q = \frac{-6+2m}{2}$ 1 $\qquad$ $= \frac{3-m}{2}$

$= -3+m$

$\therefore$ $p+q+2pq=0$

$(-3+m)+\cancel{2}\left(\frac{3-m}{\cancel{2}}\right)=0$ 1

$-3+m+3-m=0$

$0=0$ As required

c) $\int x e^{x^2}\,dx$

$= \frac{1}{2}e^{x^2}+C$

d) let $\log_8 2 = y$

$2 = 8^y$

$2 = (2^3)^y$

$1 = 3y$

$y = \frac{1}{3}$ 1

$\therefore$ $\frac{1}{3} = \log_x 5$

$x^{1/3} = 5$

$x = 5^3$

$x = 125$ 1

c) $\cos 2A = \cos A$

$2\cos^2 A - 1 = \cos A$ 1

$2\cos^2 A - \cos A - 1 = 0$

let $y = \cos A$

~~$2y^2$~~

$2y^2 - y - 1 = 0$

$(2y+1)(y-1) = 0$ 1

$y = -\frac{1}{2}$ or $y = 1$

$\therefore \cos A = -\frac{1}{2}$ or $\cos A = 1$ 1

$A = 0^\circ, 120^\circ, 240^\circ, 360^\circ$

$0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

QUESTION 2

a) $y = x^3 - 4x^2 + 8x - 8$

$f(2) = 8 - 16 + 16 - 8$

$= 0$ 1

$\therefore (x-2)$ is a factor

$$\begin{array}{r}
x^2 - 2x + 4 \\
x-2 \overline{)\, x^3 - 4x^2 + 8x - 8} \\
x^3 - 2x^2 \qquad\qquad \\
\hline
-2x^2 + 8x \qquad \\
-2x^2 + 4x \qquad \\
\hline
4x - 8 \\
4x - 8 \\
\hline
0
\end{array}$$

1

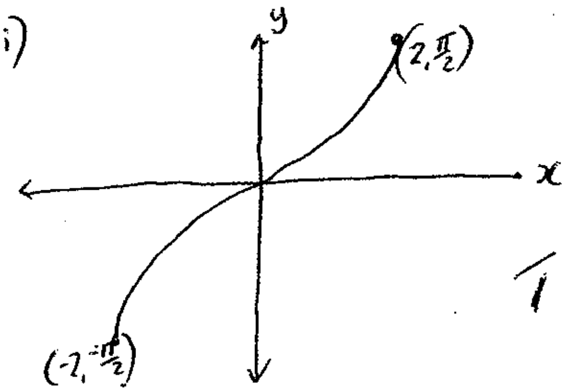
$x^3 - 4x^2 + 8x - 8 = (x-2)(x^2 - 2x + 4)$

$\Delta = 4 - 4(1)(4)$

$\Delta = -12$ No Solution

$\therefore x = 2$ 1

b) i)



1

ii) Domain $-2 \le x \le 2$ 1

Range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ 1

iii) $y = \sin^{-1}\left(\frac{x}{2}\right)$

$y' = \frac{1}{\sqrt{1 - \frac{x^2}{4}}} \cdot \frac{1}{2}$

$= \frac{1}{\sqrt{\frac{4 - x^2}{4}}} \cdot \frac{1}{2}$

$= \frac{2}{\sqrt{4 - x^2}} \cdot \frac{1}{2}$

$y' = \frac{1}{\sqrt{4 - x^2}}$ 1

when $x = 1$ $m = \frac{1}{\sqrt{4-1}}$

$= \frac{1}{\sqrt{3}}$

$= \frac{\sqrt{3}}{3}$ ½

when $x = 1$ $y = \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{6}$ ½

Question 2 (cont)

$\therefore \quad m = \frac{\sqrt{3}}{3} \qquad pt \ \left(1, \frac{\pi}{6}\right)$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\pi}{6} = \frac{\sqrt{3}}{3}(x-1)$$

$$6y - \pi = 2\sqrt{3}(x-1)$$

$$6y - \pi = \overset{2\sqrt{3}x}{\cancel{2\sqrt{x}}} - 2\sqrt{3}$$

$$2\sqrt{3}x - 6y + \pi - 2\sqrt{3} = 0$$

c) i) $x = t + 3$

$x - 3 = t$

$y = (x-3)^2 - 5$

ii) $(x-3)^2 = (y+5)$

$V = (3, -5)$

Question 3

a) i) $2\cos\left(\theta+\frac{\pi}{3}\right) = 2\left(\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right)$

$= 2\left(\cos\theta \cdot \frac{1}{2} - \sin\theta \cdot \frac{\sqrt{3}}{2}\right)$

$= \cos\theta - \sqrt{3}\sin\theta$ As required

ii) $2\cos\left(\theta+\frac{\pi}{3}\right) = 1$

$\cos\left(\theta+\frac{\pi}{3}\right) = \frac{1}{2}$

$\theta+\frac{\pi}{3} = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ or $\frac{7\pi}{3}$

$\theta = 0$ or $\frac{4\pi}{3}$ or 2π

b) $u = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$du = \frac{1}{2\sqrt{x}}\,dx \quad 2x^{1/2}du = dx$

$\int \frac{2x^{1/2}\,du}{x^{1/2} \cdot \quad 1+u^2}$

$= \int_1^3 \frac{2}{1+u^2}\,du$

$= 2\left[\tan^{-1}u\right]_1^{\sqrt{3}}$

$= 2\left(\tan^{-1}\sqrt{3}\right) - \left(\tan^{-1}1\right)$

$= 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= 2 \times \frac{\pi}{12}$

$= \frac{\pi}{6}$

c i) $x\cos^{-1}x - \sqrt{1-x^2}$

Let $t = x\cos^{-1}x$

$$\frac{dt}{dx} = x \cdot \frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x$$

$$= \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x$$

$$\therefore\ y' = \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x$$

$$y' = \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x + \frac{x}{\sqrt{1-x^2}}$$

$$y' = \cos^{-1}x.$$

ii) $\int_0^1 \cos^{-1} dx$

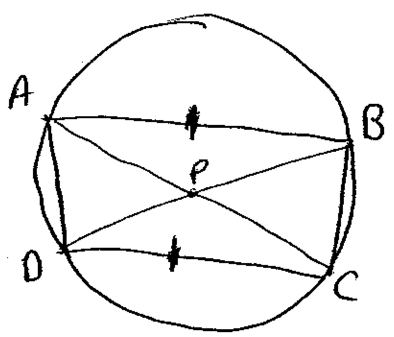
$$= \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$$

$$= \left[1\cos^{-1}1 - \sqrt{1-1^2}\right] - \left[0\cos^{-1}0 - \sqrt{1-0^2}\right]$$

$$= [0] - [0-1]$$

$$= 1.$$

d)



In Δ's ABP and DCP

$\angle ABP = \angle DCP$ ($\angle$'s in same segment)

$\angle CDP = \angle BAP$ ($\angle$'s in same segment)

$AB = DC$ (data)

$\therefore\ \Delta ABP \equiv \Delta DCP$ (ASA)

$AP = PD$ (Corresponding sides in Congruent Δ's)

$BP = PC$ (Corresponding sides in Congruent Δ's)

$AP + PC = BP + PD$

Question 4

a) $a=4 \quad r=-\frac{1}{2} \quad n=10$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{4\left(1-\left(-\frac{1}{2}\right)^{10}\right)}{1+\frac{1}{2}}$$

$$= \frac{4\left(1-\frac{1}{1024}\right)}{\frac{3}{2}}$$

$$= \frac{8}{3} \times \frac{1023}{1024}$$

$$= 2\frac{85}{128}$$

b) $\frac{a}{c} = \frac{a-B}{B-C}$

$aB - ac = ac - Bc$

$\div aBc)$ $\frac{aB}{aBc} - \frac{ac}{aBc} = \frac{ac}{aBc} - \frac{Bc}{aBC}$

$\frac{1}{c} - \frac{1}{B} = \frac{1}{B} = \frac{1}{a}$

which is proof of AP

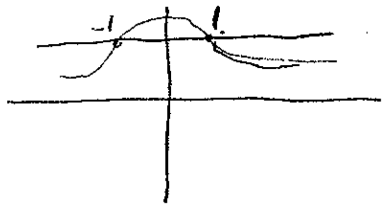
c) $\cos 2x = 2\cos^2 x - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\frac{1}{2}\left[\cos 2x + 1\right] = \cos^2 x$

$\therefore \frac{1}{2}\int \cos 2x + 1 \, dx$

$= \frac{1}{2}\left[\frac{\sin 2x}{2} + x\right] + C$

d) 

$\sqrt{2} = \sqrt{\frac{4}{1+x^2}}$

$2x^2+2=4$

$2x^2-2=0$

$x^2-1=0$

$(x-1)(x+1)=0$

$x=1, or -1$

If $y=\sqrt{\frac{4}{1+x^2}}$

$y^2 = \frac{4}{1+x^2}$

$2\pi\int_0^1 \frac{4}{1+x^2}\,dx$

$8\pi\left[\tan^{-1} x\right]_0^1$

$= 8\pi\left[\frac{\pi}{4} - 0\right]$

$= 2\pi^2$ units^3

$y=\sqrt{2}$

$y^2 = 2$

$2\pi\int_0^1 2\,dx$

$2\pi\left[2x\right]_0^1$

$2\pi\left[2-0\right]$

$= 4\pi$

$\therefore$ Volume $= 2\pi^2 - 4\pi$

$= 2\pi\left[\pi-2\right]$ units^3

Question 5

a)

When $n=1$

$1^2 = \frac{1}{6}(1+1)(2+1)$

$= \frac{1}{6} \times 2 \times 3$

$1 = 1$ $\quad$ ½

Assume true for $n=k$ and show true for $n=k+1$

$S(k) = 1^2+2^2+3^2+ \dots + k^2$

$S(k+1) = 1^2+2^2+3^2+ \dots + k^2 + (k+1)^2$ $\quad$ 1

Assume that $S(k) = \frac{1}{6}k(k+1)(2k+1)$ and show that

$S(k+1) = \frac{1}{6}(k+1)(k+2)(2k+3)$

Now $S(k+1) = S(k) + (k+1)^2$

$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$

$= (k+1)\left[\frac{1}{6}k(2k+1) + (k+1)\right]$ $\quad$ 2

$= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right)$

$= \frac{1}{6}(k+1)(2k^2+k+6k+6)$

$= \frac{1}{6}(k+1)(2k^2+7k+6)$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$

If the result is true for $n=k$, then it is true for $n=k+1$ $\quad$ ½

b) i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

$y' = \frac{x}{2a}$

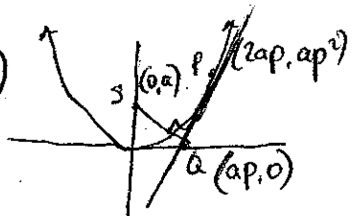
when $x = 2ap$ $\quad m = \frac{2ap}{2a}$

$m = p$ $\quad$ 1

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$ $\quad$ 2

$y = px - ap^2$ $\quad$ 1

ii) 

Co-ords of Q

$y = px - ap^2$

$0 = px - ap^2$

$xp = ap^2$

$x = ap$ $\quad$ 1

$\therefore Q(ap, 0)$

Gradient of $PQ = p$ $\quad$ ½

Gradient of $SQ = \frac{a - 0}{0 - ap}$

$= \frac{a}{-ap}$

$= \frac{-1}{p}$ $\quad$ ½

For Right Angle $\quad m_1 \times m_2 = -1$

$p \times -\frac{1}{p} = -1$

$-1 = -1$

So PQ is Perp to SQ $\quad$ 1

iii) Since $P\hat{Q}S$ is a Right Angle $\therefore$ A Semi-Circle can be drawn through P, Q and S, Since the angle in a semicircle is a right angle $\therefore$ PS is a Diameter to the circle and so the midpoint of PS must be the centre of the circle $\quad$ 1

iv) Co-ords of Midpoint of PS.

$x = \frac{0 + 2ap}{2}$ $\qquad$ $y = \frac{a + ap^2}{2}$

$x = ap$ $\qquad$ $= \frac{a(1 + p^2)}{2}$

From $x = ap$

$\frac{x}{a} = p$ $\quad$ 1

$\therefore \quad y = \left[a\left(1 + \frac{x}{a}\right)^2\right] \div 2$

$= a\left(1 + \frac{x^2}{a^2}\right) \div 2$

$= \left(a + \frac{x^2}{a}\right) \div 2$

$= \frac{a^2 + x^2}{a} \div 2$

$y = \frac{a^2 + x^2}{2a}$ $\quad$ 1

$2ay = a^2 + x^2$ $\quad$ As required