Examination Number:
Set:

Shore

Year 12
HSC Assessment Task 3
Term II Examination
2016

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- The BOSTES Reference Sheet is provided separately
- Answer Questions 1–10 on the Multiple Choice Answer Sheet provided
- In Questions 11–14 show relevant mathematical reasoning and/or calculations
- Start each of Questions 11–14 in a new writing booklet
- Write your examination number on the front cover of each booklet

If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

Total marks – 70

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–11

60 marks

- Attempt Questions 11–14
- Allow about 1 hour 45 minutes for this section

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple Choice Answer Sheet for Questions 1–10.

1 The point $P(x,y)$ divides the interval joining $A(-3,4)$ to $B(7,9)$ internally in the ratio $2:3$.

What are the coordinates of P ?

(A) $(1,6)$

(B) $(1,5)$

(C) $(0,6)$

(D) $(0,5)$

2 For what value of b is $(x+1)$ a factor of x^3-bx+3 ?

(A) $b=3$

(B) $b=-3$

(C) $b=2$

(D) $b=-2$

3 What is the acute angle between the lines $x+y=1$ and $2x-y=3$?

(A) 18°

(B) 19°

(C) 71°

(D) 72°

4 The point $P(x, y)$ moves so that it is equidistant from the y-axis and the point $S(2,0)$.

What is the equation of the locus of P ?

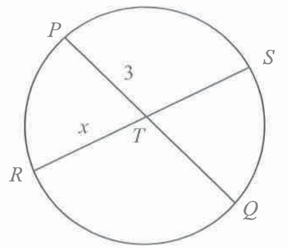
(A) $y^2 = 4(x-2)$

(B) $y^2 = 4(x-1)$

(C) $y^2 = 8(x-2)$

(D) $y^2 = 8(x-1)$

5



NOT TO SCALE

In the circle, chords PQ and RS intersect at T, $PQ = 7$, $PT = 3$, $RS = 8$ and $RT = x$.

What are the possible values of x ?

(A) $x = 6$ or $x = 2$

(B) $x = 4$ or $x = 2$

(C) $x = 6$ or $x = 4$

(D) $x = 4$ or $x = 3$

6 What is the domain for $f(x) = \sqrt{1-x} + \sqrt{1+x}$?

(A) $x > 1$ or $x < -1$

(B) $x \geq 1$ or $x \leq -1$

(C) $-1 < x < 1$

(D) $-1 \leq x \leq 1$

7 A curve has parametric equations $x = 3\cos t$ and $y = 2\sin t$.

What is the Cartesian equation of the curve ?

(A) $2x^2 + 3y^2 = 6$

(B) $2x^2 - 3y^2 = 6$

(C) $4x^2 + 9y^2 = 36$

(D) $4x^2 - 9y^2 = 36$

8 The angle θ satisfies $\sin\theta = \frac{3}{5}$ and $\frac{\pi}{2} \leq \theta \leq \pi$.

What is the value of $\sin 2\theta$?

(A) $\frac{24}{25}$

(B) $-\frac{24}{25}$

(C) $\frac{12}{25}$

(D) $-\frac{12}{25}$

9 Which one of the following is the primitive (indefinite integral) of $\frac{1}{4x^2+16}$?

(A) $\frac{1}{8}\tan^{-1}\frac{x}{2} + C$

(B) $\frac{1}{4}\tan^{-1}\frac{x}{2} + C$

(C) $\frac{1}{2}\tan^{-1} 2x + C$

(D) $\frac{1}{4}\tan^{-1} 2x + C$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section

Start each of Questions 11–14 in a new writing booklet.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Evaluate $\lim_{x \to 0} \frac{\sin x}{3x}$. **1**

(b) Solve $\frac{2x}{1+x} \le 1$. **2**

(c) Find $\int \cos^2 x \, dx$. **2**

(d) By using the substitution $\tan\frac{A}{2} = t$, or otherwise, show that **2**

$$\frac{1+\sin A+\cos A}{1+\sin A-\cos A} = \cot\frac{A}{2}.$$

(e) Find $\frac{d}{dx}\left[\cos^{-1}\left(x^2\right)\right]$. **2**

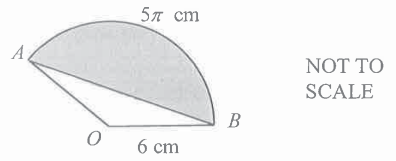
(f) (i) Write down the domain and range of the function $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$. **2**

(ii) Sketch the graph of $y = f(x)$. **1**

(g) Use the substitution $u = x^2 - 1$ to evaluate $\int_2^3 \frac{x}{\left(x^2-1\right)^2}\, dx$. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) **3**



AOB is a sector of a circle with centre O.
The length of the arc AB is 5π centimetres.

Find the exact area of the segment cut off by the arc AB and the chord AB.

(b) Simplify $\frac{4\sin 2\theta \cos 2\theta}{\cos 4\theta}$. **2**

(c) (i) Express $\sin x + \sqrt{3}\cos x$ in the form $R\sin(x+\theta)$ where $R > 0$ and $0 < \theta < \frac{\pi}{2}$. **2**

(ii) Hence or otherwise solve the equation $\sin x + \sqrt{3}\cos x = \sqrt{3}$ for $0 \le x \le 2\pi$. **2**

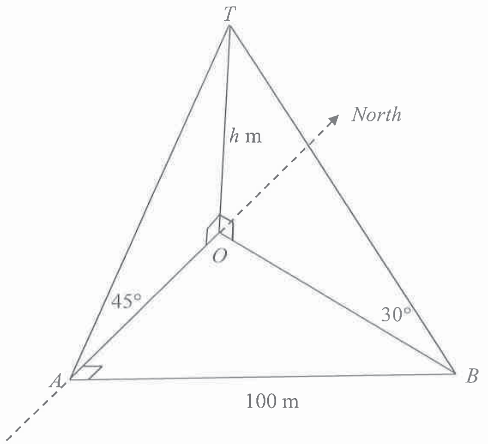
(d) Find the general solutions to $\cos\left(2x - \frac{\pi}{3}\right) = 0$. **3**

(e) Prove by mathematical induction that for all integer values of $n \ge 1$, **3**

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots\ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Question 13 **(15 marks)** Use a SEPARATE writing booklet

(a)



A surveyor stands at a point A which is due south of a tower OT of height h metres. The angle of elevation of the top of the tower from A is $45°$. The surveyor then walks due east for 100 metres to a point B, where he observes the top of the tower to have an angle of elevation of $30°$.

(i) Show that $OB = h\sqrt{3}$. **1**

(ii) Show that $h = 50\sqrt{2}$. **2**

(iii) Find the bearing of B from the base of the tower. Give your answer correct to the nearest degree. **2**

(b) The polynomial $P(x) = x^3 + px^2 + qx + r$ has roots $-\sqrt{\alpha}$, $\sqrt{\alpha}$ and β.

(i) Show that $\beta + p = 0$. **1**

(ii) Show that $\alpha\beta = r$. **1**

(iii) Show that $pq = r$. **2**

Question 13 continues on page 9

(c) The points $P(8p, 4p^2)$ and $Q(8q, 4q^2)$ lie on the parabola $x^2 = 16y$.

(i) Derive the equation of the chord PQ. Write your answer in general form. **3**

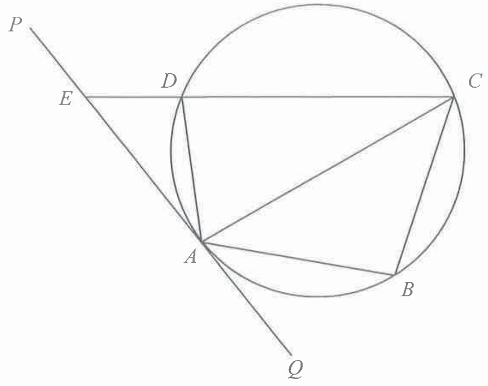
(ii) Show that if PQ is a focal chord then $pq = -1$. **1**

(iii) If PQ is a focal chord and P has coordinates $(4, 1)$, what are the coordinates of Q? **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) In the diagram, PQ is a tangent to the circle at A. Points B, C and D lie on the circle. **2**



NOT TO SCALE

Copy or trace the diagram into your writing booklet.

Show, giving reasons, that $\angle EAC = \angle ADE$.

(b) (i) Find the range of the function $y = \frac{1}{\sqrt{4-x^2}}$. **1**

(ii) Find the exact area between the curve $y = \frac{1}{\sqrt{4-x^2}}$, the x axis and the lines $x = -\sqrt{3}$ and $x = \sqrt{2}$. **2**

(c) Let $f(x) = \sin^{-1}(-x) + \cos^{-1}(-x)$, where $-1 \le x \le 1$.

(i) Show that $f'(x) = 0$. **2**

(ii) Hence deduce that $\sin^{-1}(-x) + \cos^{-1}(-x) = \frac{\pi}{2}$. **2**

Question 14 continues on page 11

(d) Given $f(x) = \log_e\left(\frac{2-x}{x}\right)$ for $0 < x < 2$.

(i) Find $f'(x)$. Express your answer as a single fraction. **2**

(ii) Explain why $f(x)$ has an inverse function for $0 < x < 2$. **2**

(iii) Find the equation of the inverse function, $f^{-1}(x)$. **2**

END OF PAPER

YEAR 12 EXTENSION 1 MID-YEAR EXAM SOLUTIONS

① $P(x,y) = \left(\frac{kx_2 + lx_1}{k+l}, \frac{ky_2+ly_1}{k+l}\right)$

$= \left(\frac{2(7)+3(-3)}{2+3}, \frac{2(9)+3(4)}{2+3}\right)$

$= \left(\frac{5}{5}, \frac{30}{5}\right)$

$= \underline{\underline{(1,6)}}$ (A)

② $P(-1) = 0 = (-1)^3 - b(-1) + 3$

$0 = -1 + b + 3$

$\underline{\underline{b = -2}}$ (D)

③ $\tan\alpha = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$

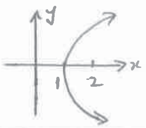
$= \left|\frac{-1-2}{1+(-1)(2)}\right|$

$= \left|\frac{-3}{-1}\right|$

$= 3$

$\alpha = \tan^{-1} 3$

$= \underline{\underline{72^\circ}}$ (D)

④ 

$\underline{\underline{y^2 = 4(x-1)}}$ (B)

⑤ $PT.TQ = RT.RS$

$3 \times 4 = x(8-x)$

$12 = 8x - x^2$

$x^2 - 8x + 12 = 0$

$(x-6)(x-2) = 0$

$\underline{\underline{x=6}}$ or $\underline{\underline{x=2}}$ (A)

⑥ $f(x) = \sqrt{1-x} + \sqrt{1+x}$

$1 - x \geq 0$ and $1 + x \geq 0$

$x \leq 1$ and $x \geq -1$

ie. $\underline{\underline{-1 \leq x \leq 1}}$ (D)

⑦ $x = 3\cos t \qquad y = 2\sin t$

$\frac{x}{3} = \cos t \qquad \frac{y}{2} = \sin t$

$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$\underline{\underline{4x^2 + 9y^2 = 36}}$ (C)

⑧ $\sin\theta = \frac{3}{5} \qquad \cos\theta = \frac{-4}{5}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \times \frac{3}{5} \times -\frac{4}{5}$

$= \underline{\underline{\frac{-24}{25}}}$ (B)

⑨ $\int \frac{1}{4x^2+16}\,dx = \frac{1}{4}\int \frac{1}{x^2+4}\,dx$

$= \frac{1}{4} \times \frac{1}{2} \tan^{-1}\frac{x}{2} + C$

(A) $= \underline{\underline{\frac{1}{8}\tan^{-1}\frac{x}{2} + C}}$

⑩ Odd function in 2nd quadrant

$\therefore \; y = \frac{-x}{1+x^2}$ (D)

A D D B A D C B A D

Question 11

(a) $\lim_{x\to 0} \frac{\sin x}{3x} = \lim_{x \to 0} \frac{\sin x}{x} \times \frac{1}{3}$

$= 1 \times \frac{1}{3}$

$= \underline{\underline{\frac{1}{3}}}$ [1]

(b) $\frac{2x}{1+x} \leq 1$

Critical Points: $x \neq -1$, $\frac{2x}{1+x} = 1$

$2x = 1 + x$

$\underline{x = 1}$

$\therefore \underline{\underline{-1 < x \leq 1}}$ [2]

(c) $\int \cos^2 x \, dx = \int \frac{1}{2}(\cos 2x + 1)\,dx$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right) + C$

[2] $= \underline{\underline{\frac{1}{4}\sin 2x + \frac{x}{2} + C}}$

(d) $LHS = \frac{1+\sin A + \cos A}{1 + \sin A - \cos A}$

$= \dfrac{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}$

$= \frac{1+t^2+2t+1-t^2}{1+t^2+2t-(1-t^2)}$

$= \frac{2t+2}{2t^2+2t}$

$= \frac{2(t+1)}{2t(t+1)}$ [2]

$= \frac{1}{t}$

$= \cot\frac{A}{2} \quad = \underline{\underline{RHS}}$

(e) $\frac{d}{dx}\left[\cos^{-1}(x^2)\right]$

$= \frac{-1}{\sqrt{1-(x^2)^2}} \times 2x$

$= \underline{\underline{\frac{-2x}{\sqrt{1-x^4}}}}$ [2]

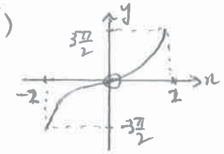
(f) $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$

(i) D: $-1 \leq \frac{x}{2} \leq 1$

$\underline{\underline{-2 \leq x \leq 2}}$ [2]

R: $-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{x}{2}\right) \leq \frac{\pi}{2}$

$\underline{\underline{-\frac{3\pi}{2} \leq 3\sin^{-1}\left(\frac{x}{2}\right) \leq \frac{3\pi}{2}}}$

(ii) 

[1]

(g) $\int_2^3 \frac{x}{(x^2-1)^2}\,dx \qquad u = x^2 - 1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\underline{x=3} \rightarrow u = 8$

$\underline{x=2} \rightarrow u = 3$

$= \int_3^8 \frac{du}{2u^2}$

$= \frac{1}{2}\left[\frac{u^{-1}}{-1}\right]_3^8$

$= -\frac{1}{2}\left[\frac{1}{u}\right]_3^8$

$= -\frac{1}{2}\left[\frac{1}{8} - \frac{1}{3}\right]$

$= -\frac{1}{2} \times \frac{-5}{24}$

$= \underline{\underline{\frac{5}{48}}}$

Question 12

(a) $l = r\theta$

$5\pi = 6\theta$

$\theta = \frac{5\pi}{6}$

$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2}\times 6^2\left(\frac{5\pi}{6} - \sin\frac{5\pi}{6}\right)$

$= 18\left(\frac{5\pi}{6} - \frac{1}{2}\right)$

$= (15\pi - 9)\ \text{cm}^2$ [3]

(b) $\frac{4\sin 2\theta\cos 2\theta}{\cos 4\theta}$

$= \frac{2\times 2\sin 2\theta\cos 2\theta}{\cos 4\theta}$

$= \frac{2\sin 4\theta}{\cos 4\theta}$ [2]

$= 2\tan 4\theta$

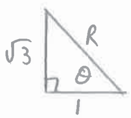
(c)

(i) $\sin x + \sqrt{3}\cos x = R\sin(x+\theta)$

$= R\sin x\cos\theta + R\cos x\sin\theta$

$\therefore R\cos\theta = 1$ $\quad R\sin\theta = \sqrt{3}$

$\cos\theta = \frac{1}{R}$ $\quad \sin\theta = \frac{\sqrt{3}}{R}$



$R^2 = 1^2 + (\sqrt{3})^2$

$= 1+3$

$= 4$

$R = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$\therefore \sin x + \sqrt{3}\cos x = 2\sin\left(x + \frac{\pi}{3}\right)$

[2]

(ii) $2\sin\left(x+\frac{\pi}{3}\right) = \sqrt{3}$

$\sin\left(x+\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \ldots\ldots$

for $0 \le x \le 2\pi$ $\quad x = 0, \frac{\pi}{3}, 2\pi$ [2]

(d) $\cos\left(2x - \frac{\pi}{3}\right) = 0$

$2x - \frac{\pi}{3} = 2n\pi \pm \cos^{-1}0$ (where n is an integer)

$2x - \frac{\pi}{3} = 2n\pi + \frac{\pi}{2}$ $\quad 2x - \frac{\pi}{3} = 2n\pi - \frac{\pi}{2}$

$2x = 2n\pi + \frac{5\pi}{6}$ $\quad 2x = 2n\pi - \frac{\pi}{6}$

$x = n\pi + \frac{5\pi}{12}$ $\quad x = n\pi - \frac{\pi}{12}$ [3]

(e) $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

Prove true for n=1

$LHS = \frac{1}{1\times 3}$ $\quad RHS = \frac{1}{2\times 1+1}$

$= \frac{1}{3}$ $\quad = \frac{1}{3}$

$\therefore$ true for n=1

Assume true for n=k

ie. $S_k = \frac{k}{2k+1}$

Prove true for n=k+1

ie. $S_k + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{k+1}{2(k+1)+1}$

ie. $S_k + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$

Q12 (e) continued

$LHS = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$= \frac{k+1}{2k+3}$

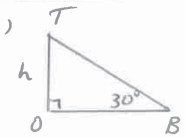
$= RHS$

$\therefore$ If true for n=k, then true for n=k+1.

By process of induction, true for all integers, $n \ge 1$. [3]

Question 13

(a) (i)



$\frac{OB}{h} = \tan 60°$

$OB = h\tan 60°$

$= h\sqrt{3}$ [1]

(ii) In $\triangle OAT$, $OA = h$

In $\triangle OAB$, $OA^2 + 100^2 = OB^2$

$h^2 + 10000 = (h\sqrt{3})^2$

$h^2 + 10000 = 3h^2$

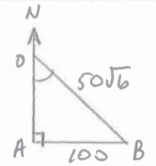
$10000 = 2h^2$

$h^2 = 5000$

$h = 50\sqrt{2}$

[2]

(iii)



$OB = h\sqrt{3}$

$= 50\sqrt{2}\times\sqrt{3}$

$= 50\sqrt{6}$

$\sin\angle AOB = \frac{100}{50\sqrt{6}}$

$= \frac{2}{\sqrt{6}}$

$\angle AOB = 55°$ [2]

$\therefore$ Bearing is $180° - 55° = 125°$

(b) (i) $-\sqrt{\alpha} + \sqrt{\alpha} + \beta = \frac{-p}{1}$

$\beta = -p$

$\beta + p = 0$ [1]

(ii) $-\sqrt{\alpha}.\sqrt{\alpha}.\beta = \frac{-r}{1}$

$-\alpha\beta = -r$

$\alpha\beta = r$ [1]

(iii) $-\sqrt{\alpha}.\sqrt{\alpha} + -\sqrt{\alpha}.\beta + \sqrt{\alpha}.\beta = q$

$-\alpha = q$

$\alpha = -q$

sub into (ii) $\quad -q\beta = r$

$\beta = \frac{r}{-q}$

sub into (i) $\quad \frac{r}{-q} + p = 0$

$r - pq = 0$

$pq = r$

[2]

(c) (i) $m = \dfrac{4p^2 - 4q^2}{8p - 8q}$

$= \dfrac{4(p-q)(p+q)}{8(p-q)}$

$= \dfrac{p+q}{2}$

Eq'n of PQ is:

$y - 4q^2 = \dfrac{p+q}{2}(x - 8q)$

$y - 4q^2 = \left(\dfrac{p+q}{2}\right)x - 4pq - 4q^2$

$\left(\dfrac{p+q}{2}\right)x - y - 4pq = 0$ [3]

(ii) PQ passes through $(0, 4)$

$\therefore \left(\dfrac{p+q}{2}\right)\times 0 - 4 - 4pq = 0$

$-4pq = 4$

$pq = -1$ [1]

(iii) $P(8p, 4p^2) \equiv (4, 1)$

$\therefore 8p = 4$

$p = \dfrac{1}{2} \rightarrow q = -2$ $(pq = -1)$

$\therefore$ Q is $(8\times -2, 4(-2)^2)$

$= (-16, 16)$ [2]

Question 14

(a) $\angle EAC = \angle ABC$

($\angle$ between chord + tangent equals $\angle$ in alternate segment)

$\angle ABC = \angle ADE$

($\angle$ in cyclic quadrilateral equals opposite exterior angle)

$\therefore \angle EAC = \angle ADE$ (both $= \angle ABC$) [2]

(b) (i) Range: $4 - x^2 \leq 4$

$0 \leq \sqrt{4-x^2} \leq 2$

$\dfrac{1}{\sqrt{4-x^2}} \geq \dfrac{1}{2}$ [1]

(ii) $A = \displaystyle\int_{-\sqrt{3}}^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}\,dx$

$= \left[\sin^{-1}\dfrac{x}{2}\right]_{-\sqrt{3}}^{\sqrt{2}}$

$= \sin^{-1}\left(\dfrac{\sqrt{2}}{2}\right) - \sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

$= \dfrac{\pi}{4} - -\dfrac{\pi}{3}$

$= \dfrac{7\pi}{12}$ units^2 [2]

(c) (i) $f(x) = \sin^{-1}(-x) + \cos^{-1}(-x)$

$f'(x) = \dfrac{1}{\sqrt{1-(-x)^2}} \times -1 + \dfrac{-1}{\sqrt{1-(-x)^2}} \times -1$

$= \dfrac{-1}{\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x^2}}$

$= 0$ [2]

(c) (ii) If $f'(x) = 0$

then $f(x) =$ constant

let $x = 0$ $f(0) = \sin^{-1}(0) + \cos^{-1}(0)$

$= 0 + \dfrac{\pi}{2}$

$= \dfrac{\pi}{2}$

$\therefore \sin^{-1}(-x) + \cos^{-1}(-x) = \dfrac{\pi}{2}$ [2]

(d) (i) $f(x) = \ln\left(\dfrac{2-x}{x}\right)$

$= \ln(2-x) - \ln x$

$f'(x) = \dfrac{-1}{2-x} - \dfrac{1}{x}$

$= \dfrac{-x - (2-x)}{(2-x)x}$

$= \dfrac{-2}{(2-x)x}$ [2]

(ii) For $0 < x < 2$, $(2-x)x > 0$

$\therefore \dfrac{-2}{(2-x)x} < 0$

i.e. $f'(x) < 0$

i.e. $f(x)$ is monotone decreasing

$\therefore f(x)$ has an inverse for $0 < x < 2$ [2]

(iii) let $y = \ln\left(\dfrac{2-x}{x}\right)$

Inverse $\Rightarrow x = \ln\left(\dfrac{2-y}{y}\right)$

$e^x = \dfrac{2-y}{y}$

$ye^x = 2 - y$

$ye^x + y = 2$

$y(e^x + 1) = 2$

$y = \dfrac{2}{e^x+1}$

$\therefore f^{-1}(x) = \dfrac{2}{e^x+1}$ [2]