St George Girls High School

Year 12

Mid-HSC Course Examination

2006

# Mathematics Extension 1

**General Instructions**
- Working time – 1½ hours
- Reading time – 5 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question in a **new booklet**.

**Total marks** – 75
- Attempt Questions 1 – 5
- All questions are of equal value

**Question 1** – (15 marks) – Start a new booklet **Marks**

a) Differentiate with respect to $x$

(i) $y = e^{\sin x}$ 1

(ii) $y = \log_e\left(\dfrac{\sqrt{x^2+1}}{x+3}\right)$ 2

b) (i) Write down the domain of the function $\ln(4-x)$ 1

(ii) Draw a neat sketch of the curve $y = \ln(4-x)$ 2

c) (i) Find $\dfrac{d}{dx}\left(x^3 \ln x\right)$ 1

(ii) Hence, or otherwise, find the value of $\int_1^e x^2 \ln x \, dx$ 3

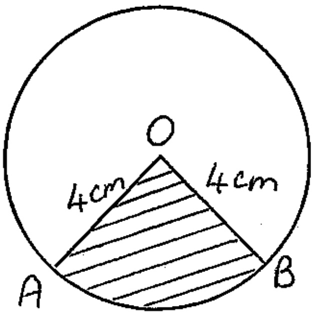
d) The value, \$$V$, of a car depreciated with time such that $\dfrac{dV}{dt} = -kV$ for some constant $k > 0$.

(i) Show that $V = Ae^{-kt}$ satisfies the differential equation. 1

(ii) After 2 years the value is \$19 808 and after a further 5 years the value is \$5135. Find the values of $k$ and $A$. 4

**Question 2** – (15 marks) – Start a new booklet **Marks**

a)




The area of minor sector $AOB$ is $20\text{cm}^2$.

(i) Find the size of $A\hat{O}B$. 2

(ii) Hence find the length of major arc $AB$. 2

b) Find the exact value of

(i) $\tan\dfrac{2\pi}{3}$ 1

(ii) $\cos 105°$ 2

c) Solve for $0 \le x \le 2\pi$

(i) $\operatorname{cosec}\left(x-\dfrac{\pi}{4}\right)=-\sqrt{2}$ 2

(ii) $\sqrt{3}\cos x = 2\sin x\,\cos x$ 3

(iii) $\tan^2 x - \sec x - 1 = 0$ 3

**Question 3** – (15 marks) – Start a new booklet **Marks**

a) Find the equation of the tangent to the curve $y = \cos 2x$ at the point where $x = \dfrac{\pi}{6}$ 3

b) Find the following:

(i) $\int \sin 3x \, dx$ 1

(ii) $\int \sin^2 3x \, dx$ 2

(iii) $\int \cos^4 x \sin x \, dx$ 2

(iv) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \, dx$ 3

c) (i) Show that the line $y = 4 - 2x$ crosses $y = \ln(x-1)$ at the point $(2, 0)$ 1

(ii) Find the acute angle between $y = 4 - 2x$ and $y = \ln(x-1)$ at the point $(2, 0)$ 3

**Question 4** – (15 marks) – Start a new booklet **Marks**

a) (i) By considering the derivative, or otherwise, show that $f(x) = \dfrac{e^x}{1+e^x}$ is an increasing function. 2

(ii) Explain why the inverse of $f(x)$ is a function. 1

(iii) Find $f^{-1}(x)$, this inverse function. 3

b) (i) Write down the domain and range of $y = 3\cos^{-1}(x+1)$ 2

(ii) Draw a neat sketch of the graph of $y = 3\cos^{-1}(x+1)$ 2

c) Find the exact value of $\tan\left(2\sin^{-1}\left(\dfrac{3}{4}\right)\right)$ 3

d) Show that $\lim\limits_{x \to 0} \dfrac{\sin 3x}{\tan 5x} = \dfrac{3}{5}$ 2

**Question 5** – (15 marks) – Start a new booklet **Marks**

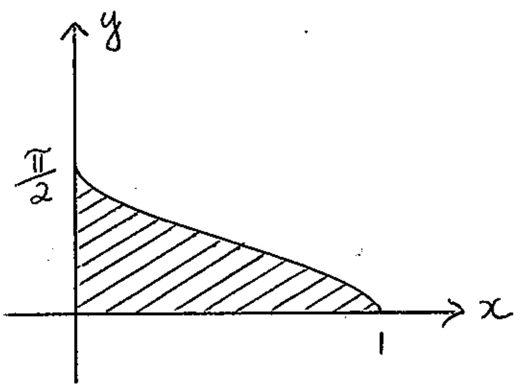
a) Find the following:

(i) $\int \frac{dx}{\sqrt{25-4x^2}}$ 2

(ii) $\int \frac{dx}{16+9x^2}$ 2

b) If $\sin x = \cos\frac{\pi}{10}$ find all possible values of $x$. 3

c) A sketch of the graph of $y = \cos^{-1}\sqrt{x}$ is shown below.




Find the shaded area. 3

(HINT: Find the area between the curve and the $y$ axis)

d) (i) Show that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}(-1 \le x \le 1)$ 2

(ii) If $\sin^{-1}A + \cos^{-1}B = \frac{11\pi}{12}$ and $\cos^{-1}A - \sin^{-1}B = \frac{5\pi}{12}$ find $A$ and $B$. 3

(HINT: Use the result from part (i))

**End of Paper**

# TABLE OF STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

*Note* $\ln x = \log_e x, \quad x > 0$

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QUESTION 1

(a) (i) $\frac{d}{dx}[e^{\sin x}] = \cos x . e^{\sin x}$

(ii) $\frac{d}{dx}\left[\log_e (x^2+1)^{\frac{1}{2}} - \log_e (x+3)\right]$

$= \frac{d}{dx}\left[\frac{1}{2}\log_e (x^2+1) - \log_e (x+3)\right]$

$= \frac{1}{2} \times \frac{2x}{x^2+1} - \frac{1}{x+3}$

$= \frac{x}{(x^2+1)} - \frac{1}{(x+3)}$

$* \quad \frac{x^2+3x-x^2+1}{(x^2+1)(x+3)}$

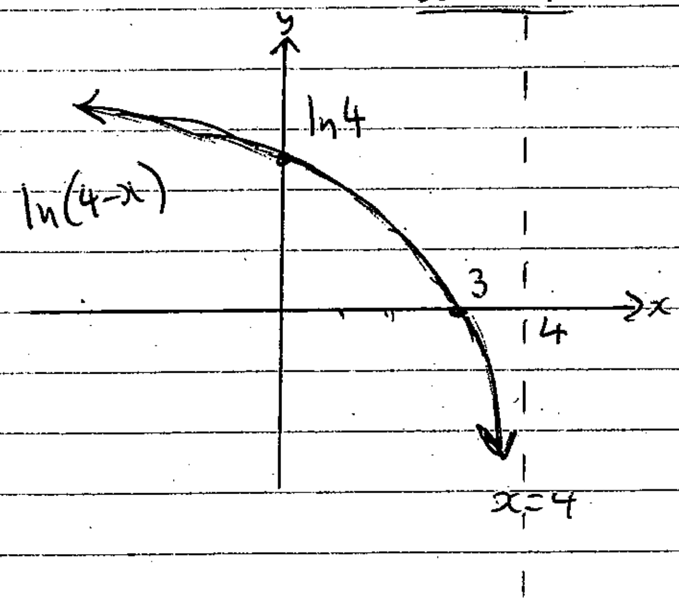
$= \frac{3x-1}{(x^2+1)(x+3)}$

(b) (i) Domain: $4-x>0$

$-x>-4$

$\underline{x<4}$

(ii)




(c) (i) $\frac{d}{dx}(x^3 . \ln x) = \ln x . 3x^2 + x^3 . \frac{1}{x}$

$= 3x^2 \ln x + x^2$

(ii) from (i) $3x^2 \ln x = \frac{d}{dx}(x^3 \ln x) - x^2$

so $x^2 \ln x = \frac{1}{3}\left[\frac{d}{dx}(x^3 \ln x) - x^2\right]$

$$\int_1^e x^2 \ln x \, dx = \frac{1}{3}\int_1^e \left[\frac{d}{dx}(x^3 \ln x) - x^2\right] dx$$

$$= \frac{1}{3}\left[x^3 \ln x - \frac{x^3}{3}\right]_1^e$$

$$= \frac{1}{3}\left[\left(e^3 \times 1 - \frac{e^3}{3}\right) - \left(0 - \frac{1}{3}\right)\right]$$

$$= \frac{1}{3}\left[e^3 - \frac{e^3}{3} + \frac{1}{3}\right]$$

$$\frac{2}{9}e^3 + \frac{1}{9}$$

(d) (i) $\frac{dV}{dt} = \frac{d}{dt}\left[Ae^{-kt}\right]$

$= -k.Ae^{-kt}$

$= -kV$ as required.

(ii) $t = 2$, $V = 19808$ & $t = 7$, $V = 5135$

$19808 = Ae^{-2k}$ ... (1) $\quad 5135 = Ae^{-7k}$ ... (2)

(1) ÷ (2) $\quad \frac{19808}{5135} = \frac{Ae^{-2k}}{Ae^{-7k}}$

$\therefore e^{5k} = 3.857...$

$5k = \ln(3.857...)$

$k = \frac{1}{5}\ln(3.857...)$

$k = 0.27$ (2 dec. pl.)

So $\frac{19808}{e^{-0.54}} = A$

Gives $A = 33991$ (nearest whole #)

QUESTION 2

(a) (i) Let $A\hat{O}B = \theta$

then $A = \frac{1}{2} r^2 \theta$

$20 = \frac{1}{2} \times 4^2 \times \theta$

$\frac{5}{2}$ rad $= \theta$

(ii) Arc length $l = r\theta$ of minor arc

$= 4 \times \frac{5}{2}$

$= 10$ cm

$\therefore$ Major arc $2\pi \times 4 - 10$

$= (8\pi - 10)$ cm

$= \underline{15.1}$ cm

(b) (i) $\tan \frac{2\pi}{3} = -\tan \frac{\pi}{3}$

$= -\sqrt{3}$

(ii) $\cos 105^\circ = \cos(60^\circ + 45^\circ)$

$= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}}$

(c) (i) $\sin\left(x - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$0 \le x \le 2\pi$

$-\frac{\pi}{4} \le x - \frac{\pi}{4} \le 2\pi - \frac{\pi}{4}$

$-\frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{7\pi}{4}$

so $x - \frac{\pi}{4} = -\frac{\pi}{4}, \left(\pi + \frac{\pi}{4}\right), \left(2\pi - \frac{\pi}{4}\right)$

$-\frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$x = 0, \frac{6\pi}{4}, \frac{8\pi}{4}$

$= \underline{0, \frac{3\pi}{2}, 2\pi}$

(ii) $\sqrt{3}\cos x - 2\sin x \cos x = 0$

$\cos x(\sqrt{3} - 2\sin x) = 0$

let $\cos x = 0$ and $\sqrt{3} - 2\sin x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$2\sin x = \sqrt{3}$

$\sin x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{3}, \pi - \frac{\pi}{3}$

$= \frac{\pi}{3}, \frac{2\pi}{3}$

(iii) $\tan^2 x = \sec^2 x - 1$ so $\tan^2 x - \sec x - 1 = 0$

becomes: $\sec^2 x - \sec x - 2 = 0$

$(\sec x - 2)(\sec x + 1) = 0$

let $\sec x = 2$ and $\sec x = -1$

$\cos x = \frac{1}{2}$ $\cos x = -1$

$x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, \pi$

$= \frac{\pi}{3}, \frac{5\pi}{3}, \pi$

QUESTION 3

(a) $\frac{dy}{dx} = -2\sin 2x$ when $x = \frac{\pi}{6}$, $y = \cos\frac{\pi}{3}$
$= \frac{1}{2}$

and gradient $m = -2\sin\frac{\pi}{3}$
$= -\sqrt{3}$

Equation of tangent

$$y - \frac{1}{2} = -\sqrt{3}\left(x - \frac{\pi}{6}\right)$$

$$y = -\sqrt{3}.x + \frac{\pi\sqrt{3}}{6} + \frac{1}{2}$$

(b) (i) $\int \sin 3x \, .dx = -\frac{1}{3}\cos 3x + c$

(ii) $\int \sin^2 3x \, .dx = \int \frac{1}{2}(1 - \cos 6x).\,dx$

$= \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + C$

$= \frac{x}{2} - \frac{\sin 6x}{12} + C$

(iii) $\int \cos^4 x . \sin x \, dx = -\int (-\sin x)(\cos x)^4 \, dx$

$= -\frac{(\cos x)^5}{5} + c$

$= -\frac{\cos^5 x}{5} + C$

(iv) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \, dx = -\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{-\sin x}{\cos x} \, dx$

$= -\left[\ln(\cos x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= -\left[\ln\left(\cos\frac{\pi}{3}\right) - \ln\left(\cos\frac{\pi}{4}\right)\right]$

$= \frac{1}{2}\ln 2$ $\quad = -\left(\ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{\sqrt{2}}\right)\right)$

$= \ln\left(\frac{1}{\sqrt{2}}\right) - \ln\left(\frac{1}{2}\right)$

$= \ln\left(\frac{2}{\sqrt{2}}\right)$

$= \ln(2^{\frac{1}{2}})$

(c) (i) $y = 4-2x$ & $y = \ln(x-1)$

when $x = 2$   when $x = 2$

$y = 0$   $y = \ln 1$

$= 0$

$\therefore$ $(2,0)$ lies on both curves.

(ii) $y' = -2$ & $y' = \frac{1}{x-1}$

$m_1 = -2$   at $x = 2$

$m_2 = \frac{1}{2-1}$

$= 1$

If $\theta$ is acute angle between curves at $(2,0)$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{-2-1}{1+-2\times 1}\right|$$

$$= 3$$

$\therefore$ $\theta = 71^\circ 34'$ (nearest min)

QUESTION 4:

(a) (i) $f'(x) = \dfrac{(1+e^x)e^x - e^x . e^x}{(1+e^x)^2}$

$= \dfrac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2}$ Since $e^x > 0$ for all $x$

$= \dfrac{e^x}{(1+e^x)^2}$ and $(1+e^x)^2 > 0$ for all $x$

then $f'(x) > 0$ for all $x$ and function is increasing over domain.

(ii) Since $f(x)$ increasing for all $x$ any horizontal line only cuts the curve in one point. It follows that the inverse will pass vertical line test, hence, the inverse will be a function.

That is, one-to-one correspondence for $x$ and $y$ values.

(iii) $y = \dfrac{e^x}{1+e^x}$

then inverse $x = \dfrac{e^y}{1+e^y}$

$x + x.e^y = e^y$

$(x-1)e^y = -x$

$e^y = \dfrac{x}{1-x}$

then $y = \ln\left(\dfrac{x}{1-x}\right)$

Gives $f^{-1}(x) = \ln\left(\dfrac{x}{1-x}\right)$

(b) (i) Domain : $-1 \le x+1 \le 1$

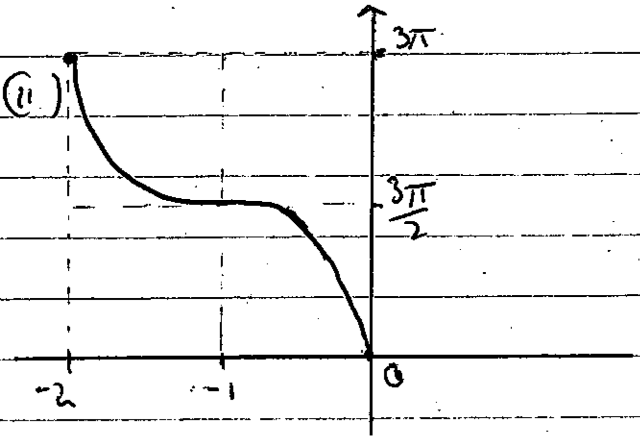
$$-2 \le x \le 0$$

Range : $0 \le \frac{y}{3} \le \pi$

$$0 \le y \le 3\pi$$

$$\left[\begin{array}{l} 0 \le \cos^{-1} f(x) \le \pi \\ \therefore 0 \le 3\cos^{-1} f(x) \le 3\pi \end{array}\right]$$

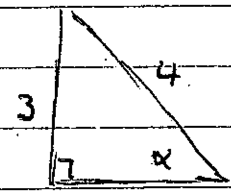
(ii)




(c) $\tan\left(2\sin^{-1}\left(\frac{3}{4}\right)\right)$

let $\sin^{-1}\left(\frac{3}{4}\right) = \alpha$

then $\frac{3}{4} = \sin\alpha$




$= \sqrt{7}$

So $\tan\alpha = \frac{3}{\sqrt{7}}$

Now $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

$$= \frac{2 \times \frac{3}{\sqrt{7}}}{1 - \left(\frac{3}{\sqrt{7}}\right)^2}$$

$$= \frac{\cancel{6}}{\sqrt{7}} \times \frac{-7}{\cancel{2}}$$

$$= -3\sqrt{7}$$

(d) $$\lim_{x\to 0} \frac{\sin 3x}{\tan 5x} = \lim_{x\to 0}\left(\frac{\sin 3x}{3x} \times \frac{5x}{\tan 5x}\right) \times \frac{3}{5}$$

$$= \frac{3}{5} \times \lim_{x\to 0}\frac{\sin 3x}{3x} \times \lim_{x\to 0}\frac{5x}{\tan 5x}$$

$$= \frac{3}{5} \times 1 \times 1$$

$$= \frac{3}{5}$$

Question 5:

(a) (i) $\displaystyle\int \frac{dx}{2\sqrt{\frac{25}{4}-x^2}} = \frac{1}{2}\sin^{-1}\frac{2x}{5} + C$ or $-\frac{1}{2}\cos^{-1}\frac{2x}{5} + C$

[use $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$]

(ii) $\displaystyle\int \frac{dx}{9\left(\frac{16}{9}+x^2\right)} = \frac{1}{9} \times \frac{1}{\frac{4}{3}} \times \tan^{-1}\frac{x}{\frac{4}{3}} + C$

$= \frac{1}{12}\tan^{-1}\frac{3x}{4} + C$

(b) Now $\cos\left(\frac{\pi}{10}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{10}\right)$

$\therefore \sin x = \sin\left(\frac{\pi}{2} - \frac{\pi}{10}\right)$

$\sin x = \sin\frac{2\pi}{5}$

$x = \frac{2\pi}{5}, \pi - \frac{2\pi}{5}, 2\pi + \frac{2\pi}{5} \ldots \quad n \in \mathbb{Z}$

$= k\pi + (-1)^k \frac{2\pi}{5} \quad k \in \mathbb{Z}$

(c) $y = \cos^{-1}\sqrt{x}$

$\cos y = \sqrt{x}$

$\cos^2 y = x$

So area bounded by curve and y-axis

$A = \displaystyle\int_0^{\frac{\pi}{2}} (\cos^2 y)\, dy$

$= \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2}\left[\cos 2y + 1\right] dy$

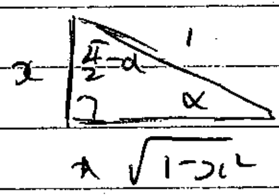
$= \frac{1}{2}\left[\frac{1}{2}\sin 2y + y\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(\left(\frac{1}{2}\sin\pi + \frac{\pi}{2}\right) - \left(\frac{1}{2}\sin 0 + 0\right)\right)$

$= \frac{\pi}{4}$ units.

(d) (i) let $\sin^{-1} x = \alpha$

$x = \sin \alpha$




then $x = \cos\left(\frac{\pi}{2} - \alpha\right)$

So

$\cos^{-1} x = \frac{\pi}{2} - \alpha$

Gives $\sin^{-1} x + \cos^{-1} x = \alpha + \frac{\pi}{2} - \alpha$

$= \frac{\pi}{2}$

or $\frac{d}{dx}\left[\sin^{-1} x + \cos^{-1} x\right] = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$

$= 0$

$\therefore \sin^{-1} x + \cos^{-1} x = c$ ($c$ is constant

Subst $x = \frac{1}{2}$

$y = \sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{6} + \frac{\pi}{3}$

$= \frac{3\pi}{6}$

$= \frac{\pi}{2}$

$c = \frac{\pi}{2}$

(ii) From (i) $\sin^{-1} A = \frac{\pi}{2} - \cos^{-1} A$

$\cos^{-1} B = \frac{\pi}{2} - \sin^{-1} B$

So $\sin^{-1} A + \cos^{-1} B = \frac{11\pi}{12}$ - - - ⊛

is $\left(\frac{\pi}{2} - \cos^{-1} A\right) + \left(\frac{\pi}{2} - \sin^{-1} B\right) = \frac{11\pi}{12}$

$-\cos^{-1} A - \sin^{-1} B = -\frac{\pi}{12}$

$\cos^{-1} A + \sin^{-1} B = \frac{\pi}{12}$ - - ①

$$\cos^{-1}A - \sin^{-1}B = \frac{5\pi}{12} \quad \cdots \text{②}$$

$$\cos^{-1}A + \sin^{-1}B = \frac{\pi}{12}$$

①+②

$$2\cos^{-1}A = \frac{\pi}{2}$$

$$\cos^{-1}A = \frac{\pi}{4}$$

$$\therefore\ A = \cos\frac{\pi}{4}$$

$$A = \frac{1}{\sqrt{2}}$$

①-② $$-2\sin^{-1}\beta = \frac{\pi}{3}$$

$$\sin^{-1}\beta = -\frac{\pi}{6}$$

$$\beta = \sin\left(-\frac{\pi}{6}\right)$$

$$= -\sin\frac{\pi}{6}$$

$$= -\frac{1}{2}$$