St George Girls High School

Year 12

Mid-HSC Course Examination

2007

Mathematics Extension 1

General Instructions

- Working time – 1½ hours
- Reading time – 5 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question in a **new booklet.**

Total marks – 60

- Attempt Questions 1 – 5
- All questions are of equal value

Question 1 (12 marks) **Marks**

a) The growth rate per hour, $\frac{dP}{dt}$, of a population of bacteria, P, is 12% of the population at that time. Initially the population is 100 000.

(i) Show that the population in any hour can be calculated by the model: 1

$$P = P_0 e^{0.12t}$$

(ii) Sketch the curve of population against time

(iii) Determine the population after 4 hours 2

b) An amount of water, W litres, in a tank, evaporates at a rate proportional to the amount of water in the tank at that time. This can be represented by $\frac{dW}{dt} \alpha W$.

Initially the tank is full and the quantity is reduced by $\frac{1}{4}$ after 120 hours.

(i) Show that this situation can be represented by $W = W_0 e^{kt}$ 1

(ii) Find an expression for the exact value of k 2

(iii) What exact fraction of the water has evaporated after 240 hours? 1

(iv) When will there be only $\frac{1}{4}$ of the water left in the tank? 3

Question 2 (12 marks) **Marks**

a) A cubic polynomial $P(x)$ gives remainders of 1 and 2 when divided by $x + 2$ and $x - 1$ respectively. Find the remainder when it is divided by $(x + 2)(x - 1)$.
Hint: let the remainder be linear. 3

b) (i) Factorise $P(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$ completely. 4

(ii) Hence, sketch $y = x^4 - 2x^3 - 3x^2 + 8x - 4$ without using calculus, showing all intercepts. 2

c) Solve $x^3 + 6x^2 + 11x + 6 = 0$ if the roots are in arithmetic progression. 3

Question 3 (12 marks) **Marks**

a) If α, β, γ are the roots of the equation $x^3-2x+5=0$, find:

(i) $\alpha+\beta+\gamma$ 1

(ii) $\alpha\beta+\beta\gamma+\alpha\gamma$ 1

(iii) $\alpha\beta\gamma$ 1

(iv) $\alpha^2+\beta^2+\gamma^2$ 2

(v) $\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2$ 1

(vi) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}+\dfrac{1}{\gamma^2}$ 3

b) The line y = $3x$ - 2 is the tangent to the curve y = x^3 at the point (1, 1). Find the point where this tangent meets the curve again.

Question 4 (12 marks) **Marks**

a) Solve $\cos 2x - 3\cos x = 1$, for $0° \le x \le 360°$ 4

b) Find possible values of a if the lines $2x + 3y - 5 = 0$ and $ax + 2y + 3 = 0$ are inclined to each other at 45° 4

c) Sketch the graph of $y = \sin(2x - \frac{\pi}{4})$ for $0 \le x \le \pi$ 2

d) OAB is a sector of a circle, with radius 3cm and an angle of 30° subtended at the centre, O, of the circle. Find the **exact** area of the minor segment formed by the chord AB. 2

Question 5 (12 marks) **Marks**

a) Differentiate:

(i) $\log_e(\cos x)$ 1

(ii) $\sin^3(2x+1)$ 2

b) Find the equation of the tangent at $x=\frac{\pi}{3}$ on the curve $y = \tan x$ 3

c) (i) Evaluate $\int_0^{\frac{\pi}{6}} \sin 2x \, dx$ 2

(ii) Find $\int \tan^2(x+1) \, dx$ 2

d) Find the area of the region bounded by the curve $y = \sin x$ and the x-axis from $x = \frac{\pi}{2}$ to $x = -\frac{\pi}{2}$ 2

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Year 12 Mid HSC Mathematics Ext. 1 - 2007

SOLUTIONS

Question 1

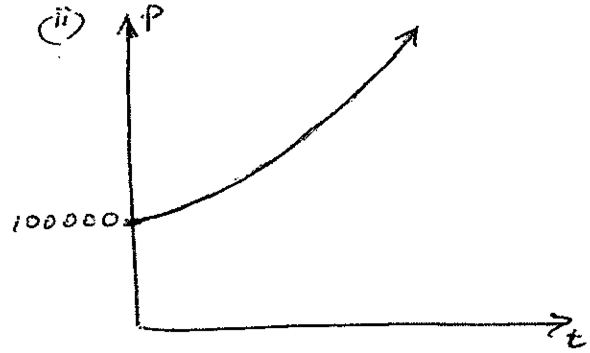
a) (i) $P = P_0 e^{0.12t}$

$\frac{dP}{dt} = 0.12 P_0 e^{0.12t}$

$= 0.12P$

$\therefore \frac{dP}{dt}$ is 12% of the population at any time t.

(ii)



(iii) $P = 100000e^{0.12 \times 4}$

$= 100\,000e^{0.48}$

$\doteqdot 161\,600$

b) (i) $W = W_0 e^{kt}$

$\frac{dW}{dt} = kW_0 e^{kt}$

$= kW$

$\therefore \frac{dW}{dt} \propto W$

(ii) when $t = 120$

$W = \frac{3}{4}W_0$

$\frac{3}{4}W_0 = W_0 e^{120k}$

$\frac{3}{4} = e^{120k}$

$120k = \log_e \frac{3}{4}$

$k = \frac{1}{120} \log_e \frac{3}{4}$

(iii) $W = W_0 e^{\frac{1}{120}\log_e \frac{3}{4} \times 240}$

$= W_0 e^{2\log_e \frac{3}{4}}$

$= W_0 e^{\log_e \frac{9}{16}}$

$= W_0 \times \frac{9}{16}$

$\therefore$ there is $\frac{9}{16}$ of the water left after 240 hours

$\therefore \frac{7}{16}$ of the water has evaporated

(iv) $\frac{1}{4}W_0 = W_0 e^{\frac{1}{120}\log_e \frac{3}{4} \times t}$

$\frac{1}{4} = e^{\frac{t}{120}\log_e \frac{3}{4}}$

$\log_e \frac{1}{4} = \frac{t}{120} \log_e \frac{3}{4}$

$120 \log_e \frac{1}{4} = t \log_e \frac{3}{4}$

$t = \frac{120 \log_e \frac{1}{4}}{\log_e \frac{3}{4}}$

$= 578.26$ hours

Question 2

a) Let $R(x)$ be the remainder

$R(x) = ax + b$

$R(-2) = 1$

$-2a + b = 1$ ——— ①

$R(1) = 2$

$a + b = 2$ ——— ②

② − ①

$3a = 1$

$a = \frac{1}{3}$

$b = \frac{5}{3}$

$\therefore$ the remainder will be

$\frac{1}{3}x + \frac{5}{3}$

b) (i) $P(1) = 0$

$P(2) = 0$

$P(-2) = 0$

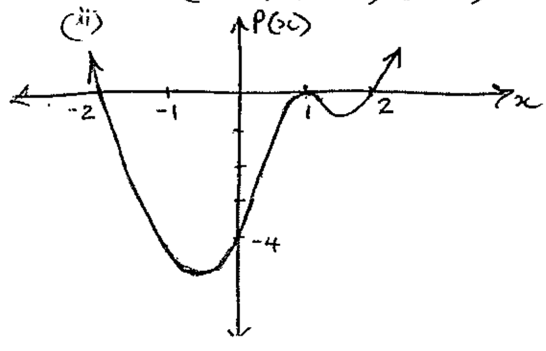
$$\begin{array}{r}
x^2 - 2x + 1 \\
x^2 - 4 \,\big)\, x^4 - 2x^3 - 3x^2 + 8x - 4 \\
x^4 \qquad - 4x^2 \qquad\qquad \\
\hline
-2x^3 + x^2 + 8x \qquad \\
-2x^3 \qquad + 8x \qquad \\
\hline
x^2 \qquad - 4 \\
x^2 \qquad - 4 \\
\hline
0
\end{array}$$

$P(x) = (x^2 - 4)(x^2 - 2x + 1)$

$= (x^2 - 4)(x - 1)^2$

$= (x - 2)(x + 2)(x - 1)^2$

(ii)



c) Let the roots be α, β and $\frac{\alpha+\beta}{2}$

$$\alpha+\beta+\frac{\alpha+\beta}{2} = -6$$
$$2\alpha+2\beta+\alpha+\beta = -12$$
$$3\alpha+3\beta = -12$$
$$\alpha+\beta = -4 \quad \text{——} ①$$

$$\alpha\beta+\alpha\left(\frac{\alpha+\beta}{2}\right)+\beta\left(\frac{\alpha+\beta}{2}\right) = 11$$
$$\alpha\beta+\alpha\times\frac{-4}{2}+\beta\times\frac{-4}{2} = 11$$
$$\alpha\beta-2\alpha-2\beta = 11$$
$$\alpha\beta-2(\alpha+\beta) = 11$$
$$\alpha\beta+8 = 11$$
$$\alpha\beta = 3 \quad \text{——} ②$$

from ①
$$\beta = -4-\alpha \quad \text{——} ③$$
sub ③ into ②
$$\alpha(-4-\alpha) = 3$$
$$-4\alpha-\alpha^2 = 3$$
$$\alpha^2+4\alpha+3 = 0$$
$$(\alpha+1)(\alpha+3) = 0$$
$\therefore$ $\alpha = -1$ or $\alpha = -3$

$\beta = -3$ $\quad\quad$ $\beta = -1$

$\therefore$ the roots are $-3, -2, -1$

i.e. $x = -3, -2, -1$

is the solution

Question 3

a) (i) $\alpha+\beta+\gamma = 0$

(ii) $\alpha\beta+\alpha\gamma+\beta\gamma = -2$

(iii) $\alpha\beta\gamma = -5$

(iv) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$
$$= 0^2 - 2\times(-2)$$
$$= 4$$

(v) $\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2$
$$= (\alpha\beta\gamma)(\alpha+\beta+\gamma)$$
$$= -5\times 0$$
$$= 0$$

(vi) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2}{(\alpha\beta\gamma)^2}$

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2 = (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)$$
$$= (-2)^2 - 2\times 0$$
$$= 4$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{4}{(-5)^2}$$
$$= \frac{4}{25}$$

b) $y = 3x-2$ —— ①

$y = x^3$ —— ②

sub. ② into ①
$$x^3 = 3x-2$$
$$x^3-3x+2 = 0$$

$(x-1)^2$ is a solution to this equa[tion]

as $y=3x-2$ is a tangent to $y=x^3$ at $x=1$

$$\begin{array}{r} x+2 \\ x^2-2x+1 \overline{)\, x^3 \qquad -3x+2} \\ \underline{x^3-2x^2+x} \\ 2x^2-4x+2 \\ \underline{2x^2-4x+2} \\ 0 \end{array}$$

$\therefore$ $(x-1)^2(x+2) = 0$

at $x=-2$ $\quad y=-8$

$\therefore$ the tangent meets the curve again at $(-2,-8)$

Question 4

a) $\cos 2x - 3\cos x = 1$
$$2\cos^2 x - 1 - 3\cos x = 1$$
$$2\cos^2 x - 3\cos x - 2 = 0$$
$$(2\cos x+1)(\cos x-2) = 0$$

either

$2\cos x+1 = 0$ $\quad$ or $\quad$ $\cos x - 2 = 0$

$\cos x = \frac{-1}{2}$ $\quad\quad$ $\cos x = 2$ no solution

$x = 120°, 240°$

b) $m_1 = \frac{-2}{3}$ $\quad$ $m_2 = \frac{-a}{2}$

$$\left|\frac{m_1-m_2}{1+m_1m_2}\right| = \tan 45°$$

$$\left|\frac{\frac{-2}{3}+\frac{a}{2}}{1+\frac{a}{3}}\right| = 1$$

either $\dfrac{-\frac{2}{3}+\frac{a}{2}}{1+\frac{a}{3}} = 1$

$-\frac{2}{3}+\frac{a}{2} = 1+\frac{a}{3}$

$\frac{a}{6} = \frac{5}{3}$

$a = 10$

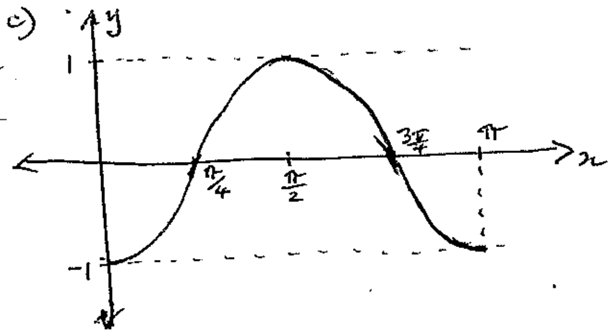
or $\dfrac{-\frac{2}{3}+\frac{a}{2}}{1+\frac{a}{3}} = -1$

$-\frac{2}{3}+\frac{a}{2} = -1-\frac{a}{3}$

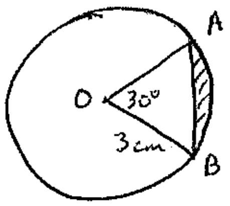
$\frac{5a}{6} = -\frac{1}{3}$

$a = -\frac{6}{15}$

c)



d)



Area $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$\theta = \frac{\pi}{6}, \ r = 3$

Area $= \frac{9\pi}{12} - \frac{9}{2}\sin\frac{\pi}{6}$

$= \frac{3\pi}{4} - \frac{9}{4}$

$= \frac{3\pi - 9}{4}$ cm^2

Question 5

a) (i) $\frac{d}{dx}\left(\log_e(\cos x)\right) = \frac{-\sin x}{\cos x}$

$= -\tan x$

(ii) $\frac{d}{dx}\left(\sin^3(2x+1)\right)$

$= 6\cos(2x+1)\sin^2(2x+1)$

b) $y = \tan x$ $\qquad \frac{dy}{dx} = \sec^2 x$

$= \sqrt{3}$ when $x = \frac{\pi}{3}$ $\qquad = 4$ when $x = \frac{\pi}{3}$

$y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$

$\therefore$ the equation of the tangent is

$4x - y + \sqrt{3} - \frac{\pi}{3} = 0$

c) (i) $\int_0^{\frac{\pi}{6}} \sin 2x \, dx = \left[\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\cos\frac{\pi}{3} - \frac{1}{2}\cos 0$

$= \frac{1}{4} - \frac{1}{2}$

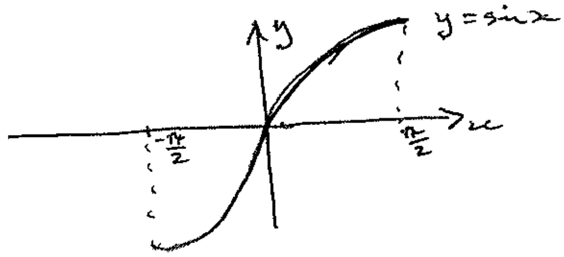
$= -\frac{1}{4}$

(ii) $\int \tan^2(x+1)\,dx = \int\left(\sec^2(x+1) - 1\right)dx$

$= \int \sec^2(x+1)\,dx - \int dx$

$= \tan(x+1) - x + C$

d)



Area $= 2\int_0^{\frac{\pi}{2}} \sin x \, dx$

$= 2\left[-\cos x\right]_0^{\frac{\pi}{2}}$

$= 2(-0 + 1)$

$= 2$ units^2