Name: ____________________

Teacher: ____________________

YEAR 12 MATHEMATICS

EXTENSION 1

HALF YEARLY EXAMINATION 2005

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working must be shown in every question
- All questions are of equal value
- Start each question on a separate page

Question 1 **Marks**

(a) Find the size of the acute angle between the lines **3**

$$y = -x$$

$$\sqrt{3}y = x$$

(b) Solve the inequality $\dfrac{2x+1}{x-1} > 3$. **2**

(c) A sector of angle $135°$ at the centre, is cut from a circular piece of cardboard of radius 8cm. The cut edges are brought together to form a cone. Find the circumference of the base of the cone. **2**

(d) For the given function $f(x) = \dfrac{8}{4+x^2}$,

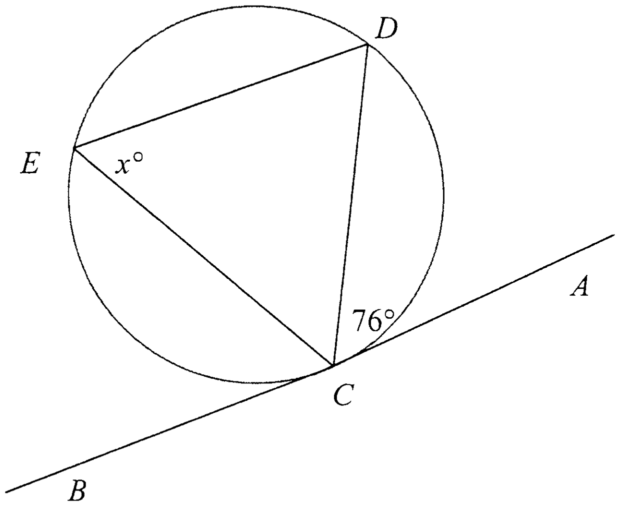
(i) show that $f(x)$ is an even function. **2**

(ii) evaluate $\lim\limits_{x \to \infty} \dfrac{8}{4+x^2}$. **2**

(iii) sketch the graph of $y = f(x)$. **1**

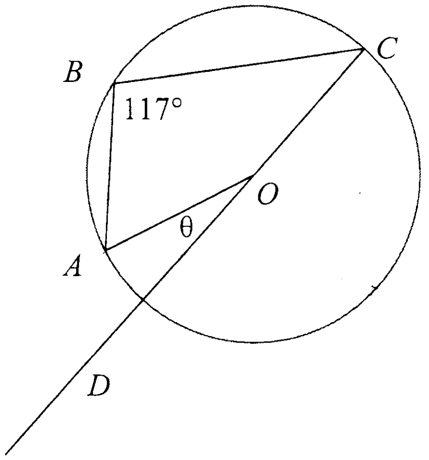
Question 2 *Start a new page* **Marks**

(a) In the diagram drawn below AB is a tangent to the circle and $\angle DCA = 76°$. Find the value of the pronumeral, giving reasons for your answer. **2**



NOT TO SCALE

(b) Given that O is the centre of the circle and $\angle ABC = 117°$, find the value of θ, giving reasons for your answer. **2**



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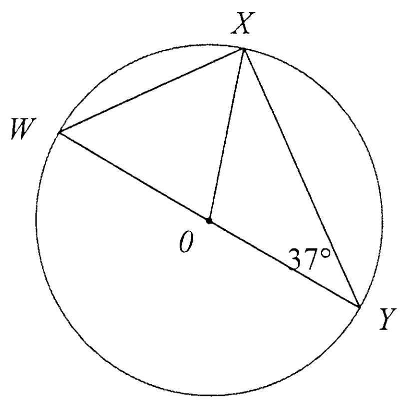
(c) In the diagram drawn below, WY is a diameter of a circle, centre O. If $\angle WYX = 37°$, find the size of $\angle WXO$. Give reasons for your answer. **2**



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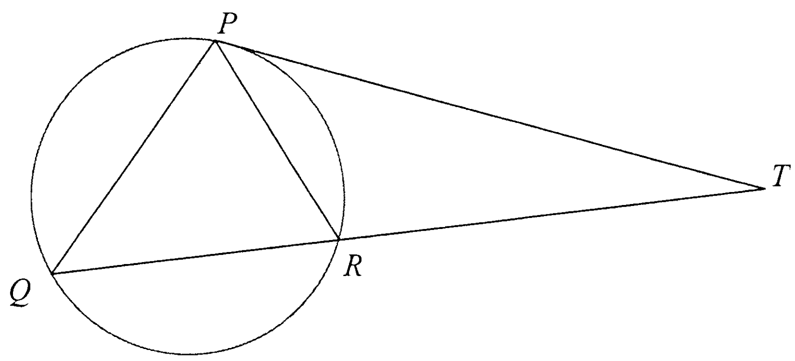
Question 2 (cont.) **Marks**

(d) PT is a tangent to the circle drawn below and QR is a secant, intersecting the circle at Q and at R. The line QR intersects PT at T.



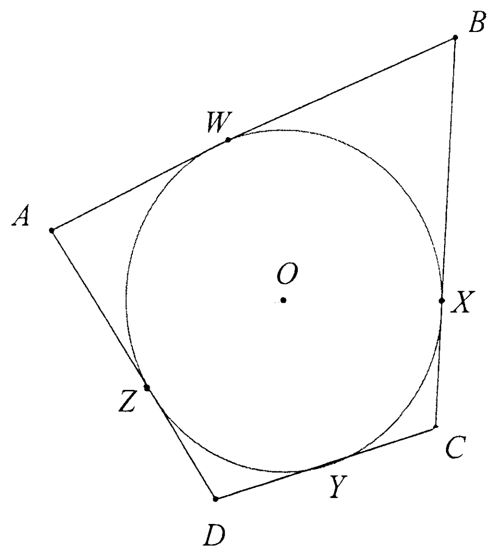
(i) Prove that the triangles PRT and QPT are similar. **2**

(ii) Hence prove $PT^2 = QT \times RT$. **1**

(e) A quadrilateral $ABCD$ is constructed so that AB, BC, CD and DA are tangents to a circle. W, X, Y and Z are the points of contact of the tangents AB, BC, CD and DA respectively.

(i) Copy the diagram neatly.

(ii) Prove that AB+DC=AD+BC. **3**



Question 3 *Start a new page* **Marks**

(a) Find the value of k for which $(x+2)$ is a factor of the polynomial **3**

$$2x^3 + kx^2 - 18x - 8.$$

Hence, express the polynomial as a product of its linear factors.

(b) Sketch the graph of the polynomial **2**

$$P(x) = x(x+1)^2(2x-1).$$

(c) If α, β, γ are the roots of the equation $x^3 + 2x^2 + 3x + 4 = 0$, **3**

find the value of (i) $\alpha + \beta + \gamma$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$

(iii) $\alpha^2 + \beta^2 + \gamma^2$.

(d) The polynomial $x^3 - 6x^2 + 9x - k$ has a double root. **4**

Show that there are two possible values of k.
Find the roots for each value of k.

Question 4 *Start a new page* **Marks**

(a) Sketch the function $y = 2\sin 3x$ for $0 \le x \le 2\pi$. **2**

(b) Differentiate $\cos^4 x$. **2**

(c) Solve $7\sin\theta + \cos\theta = 5$ for $0^\circ \le \theta \le 360^\circ$. **3**

Write your solution(s) to the nearest minute.

(d) Use the table of standard integrals to show that: **2**

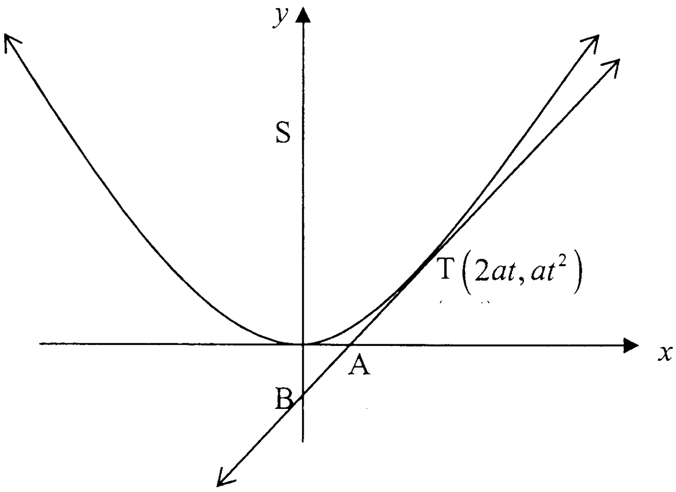
$$\int_0^{\frac{\pi}{9}} \sec 3x \tan 3x \, dx = \frac{1}{3}$$

(e) Prove that $\dfrac{1}{\tan A + \cot B} + \dfrac{1}{\cot A + \tan B} = \dfrac{\sin(A+B)}{\cos(A-B)}$. **3**

Question 5 *Start a new page* **Marks**

(a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$, are two points on the parabola $x^2 = 4ay$. If PQ is a focal chord, prove that $pq = -1$. **3**

(b) $T(2at, at^2)$ is a variable point on the parabola $x^2 = 4ay$.



(i) Show that the equation of the tangent, to the parabola at the point T, is $y - tx + at^2 = 0$. **1**

(ii) If the tangent at T cuts the x-axis at A, and the y-axis at B, find the co-ordinates of A and B. **2**

(iii) Show that the tangent at T makes equal angles with the y-axis and the line TS, where S is the focus of the parabola. **3**

(iv) In what ratio does the point T, divide the interval AB? **3**

Question 6 *Start a new page* **Marks**

(a) Solve $\log_3(9x-2)-2\log_3 x=2$. **3**

(b) Evaluate $\int_0^1 \frac{3x^2}{1+x^3}dx$. **2**

(c) Calculate the volume of the solid of revolution, **2**

formed by rotating the curve $y=e^x+e^{-x}$, about the x-axis,

between $x=-1$ and $x=1$.

(d) Consider the function $y=xe^{-x}$.

(i) Determine the nature of any stationary point(s). **2**

(ii) Find any point(s) of inflexion. **2**

(iii) Sketch the function. **1**

Question 7 *Start a new page* **Marks**

(a) Evaluate $\int_0^4 x\sqrt{16-x^2}\,dx$, 3

using the substitution $u = 16 - x^2$, or otherwise.

(b) A solid of revolution is formed by rotating the area under the curve $y = \tan x$ between $x = 0$ and $x = \frac{\pi}{3}$, around the $x -$ axis. 3

Find the exact volume of the solid.

(c) Show that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. 3

Hence, find the exact value of $\int_0^{\frac{\pi}{2}} \sin^2 \frac{\theta}{2}\,d\theta$.

(d) For a certain function, $f''(x) = -18\cos 3x$. 3

Determine the equation of this function,

given that there is a stationary point at the point $\left(\frac{2\pi}{3}, 1\right)$.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

SOLUTIONS TO YEAR 12 (EXT. 1) HALF-YEARLY 2005.

QUESTION 1.

(a) $y = -x$

$m_1 = -1$

$y = 1/\sqrt{3}\, x$

$m_2 = 1/\sqrt{3}$

ie/ $m_2 = \sqrt{3}/3$

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{-1 - \sqrt{3}/3}{1 - \sqrt{3}/3} \right|$$

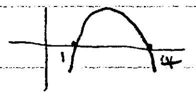
$$= \frac{3 + \sqrt{3}}{3 - \sqrt{3}}$$

$\therefore\ \theta = 75^\circ$

(b) $(2x+1)(x-1) > 3(x-1)^2$

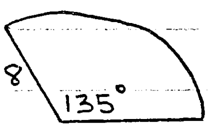
$(x-1)[2x+1 - 3(x-1)] > 0$

$(x-1)(4-x) > 0$



$\therefore\ 1 < x < 4$

(c)



$l = r\theta$

$= 8 \times \frac{3\pi}{4}$

$= 6\pi$

$\therefore$ the circumf. is 6π cm.

(d) (i) $f(-x) = \frac{8}{4 + (-x)^2}$

$= \frac{8}{4 + x^2}$

$= f(x)$

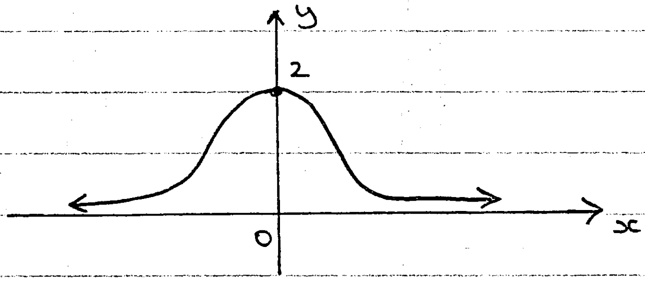
$\therefore f(x)$ is an even fn.

(ii) $\lim_{x \to \infty} \frac{8}{4 + x^2} = \lim_{x \to \infty} \frac{8/x^2}{4/x^2 + 1}$

$= \frac{0}{0+1}$

$= 0$

(iii) when $x = 0$, $y = 2$



QUESTION 2.

a) $x = 76$ (alt. seg. thm)

b) reflex $\angle AOC = 234^\circ$ (angle at centre is twice $\angle$ at circumf.)

$\theta = 234^\circ - 180^\circ$ (COD is a diam)

$\therefore \theta = 54^\circ$

c) $\angle WXY = 90^\circ$ ($\angle$ in a semi-circle)

$\angle OXY = 37^\circ$ (base $\angle$'s of isos. Δ)

$\therefore \angle WXO = 53^\circ$

d) i) In ΔPRT and ΔQPT

$\angle T$ is common

$\angle TPR = \angle TQP$ (alt. seg. thm)

$\therefore \Delta PRT \parallel\!| \Delta QPT$ (equiangular)

ii) $\dfrac{PT}{QT} = \dfrac{RT}{PT}$ (ratio of corres. sides in sim. Δ's)

$\therefore PT^2 = QT \times RT$

e) ii) $BW = BX$ (tangents to a circle, from ext. pt., are equal)

$AW = AZ$ " " " "

$DY = DZ$ " " " "

$CY = CX$ " " " "

So, $(AW + BW) + (DY + CY) = (AZ + BX) + (DZ + CX)$

ie $AB + DC = (AZ + DZ) + (BX + CX)$

$= AD + BC$

$\therefore AB + DC = AD + BC$

QUESTION 3.

(a) Let $P(x) = 2x^3 + kx^2 - 18x - 8$, then $P(-2) = 0$

$0 = -16 + 4k + 36 - 8$

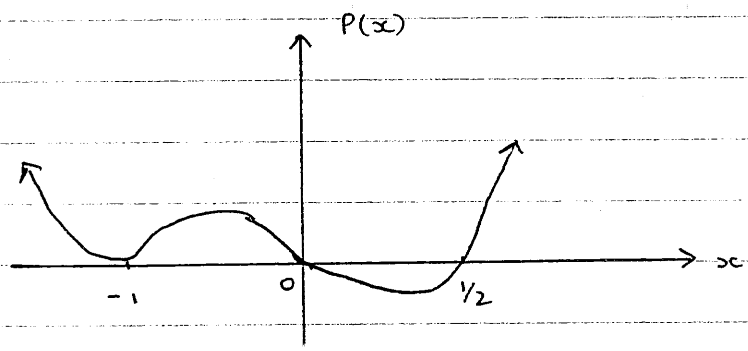
$\therefore \; k = -3$

$$
\begin{array}{r|l}
 & 2x^2 - 7x - 4 \\
\hline
x+2 & 2x^3 - 3x^2 - 18x - 8 \\
 & \underline{2x^3 + 4x^2} \\
 & \quad -7x^2 - 18x \\
 & \quad \underline{-7x^2 - 14x} \\
 & \qquad -4x - 8 \\
 & \qquad \underline{-4x - 8}
\end{array}
$$

$\therefore \; 2x^3 - 3x^2 - 18x - 8 = (x+2)(x-4)(2x+1)$

(b) $P(x) = x(x+1)^2(2x-1)$

$P(1) > 0$



(c) (i) $\alpha + \beta + \gamma = -b/a$

$= -2$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = c/a$

$= 3$

(iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 4 - 6$

$= -2$

(d) Let the roots be α, α, β

then $2\alpha + \beta = 6$ — ①

$\alpha^2 + 2\beta\alpha = 9$ — ②

$\alpha^2\beta = k$ — ③

from ① $\beta = 6 - 2\alpha$

sub. in ② $\alpha^2 + 2\alpha(6 - 2\alpha) = 9$

$$\alpha^2 + 12\alpha - 4\alpha^2 = 9$$

$$-3\alpha^2 + 12\alpha - 9 = 0$$

$$\alpha^2 - 4\alpha + 3 = 0$$

$$(\alpha - 3)(\alpha - 1) = 0$$

$$\alpha = 3, 1$$

when $\alpha = 3$, $\beta = 0$ and $K = 0$

" $\alpha = 1$, $\beta = 4$ and $K = 4$

$\therefore$ if $K = 0$, the roots are $3, 3, 0$

if $K = 4$, " " " $1, 1, 4$

QUESTION 4.

a) $$\tan A + \cot B = \frac{\sin A}{\cos A} + \frac{\cos B}{\sin B}$$

$$= \frac{\sin A \sin B + \cos A \cos B}{\cos A \sin B}$$

$$= \frac{\cos(A - B)}{\cos A \sin B}$$

$$\cot A + \tan B = \frac{\cos A}{\sin A} + \frac{\sin B}{\cos B}$$

$$= \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B}$$

$$= \frac{\cos(A - B)}{\sin A \cos B}$$

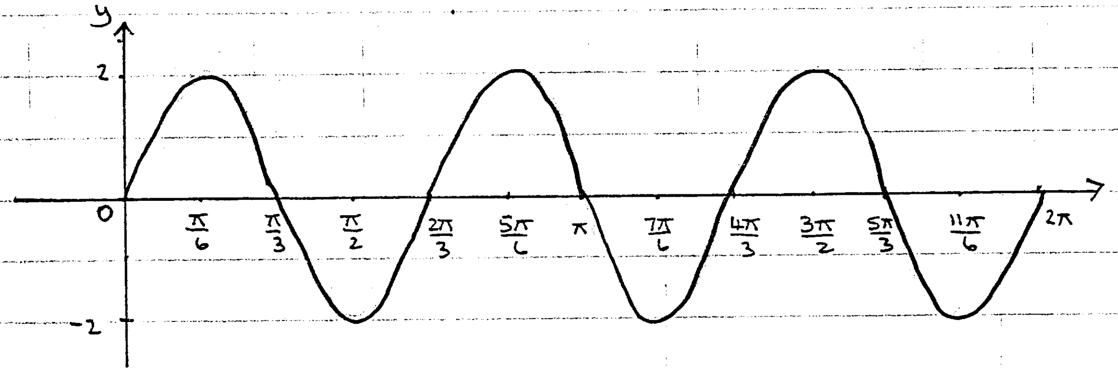
$$LHS = \frac{\cos A \sin B}{\cos(A - B)} + \frac{\sin A \cos B}{\cos(A - B)}$$

$$= \frac{\cos A \sin B + \sin A \cos B}{\cos(A - B)}$$

$$= \frac{\sin(A + B)}{\cos(A - B)}$$

$$= RHS.$$

(a)



(b) Let $y = \cos^4 x$

$$\frac{dy}{dx} = 4\cos^3 x \times -\sin x$$

ie/ $$\frac{dy}{dx} = -4\sin x\cos^3 x$$

(c) $7\sin\theta + \cos\theta = R\sin(\theta + \alpha)$

$R = \sqrt{50}$

ie $R = 5\sqrt{2}$

$\tan\alpha = 1/7$

$\alpha = 8^\circ 8'$

$5\sqrt{2}\sin(\theta + 8^\circ 8') = 5$ $\qquad 0^\circ \leq \theta \leq 360^\circ$

$\sin(\theta + 8^\circ 8') = 1/\sqrt{2}$ $\qquad 8^\circ 8' \leq \theta \leq 368^\circ 8'$

✓ ✓

$\theta + 8^\circ 8' = 45^\circ$ or 135°

$\theta = 36^\circ 52'$ or $126^\circ 52'$

or//

$7 \cdot \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 5$ $\qquad 0^\circ \leq \frac{\theta}{2} \leq 180^\circ$

$14t + 1 - t^2 = 5 + 5t^2$

$6t^2 - 14t + 4 = 0$

$3t^2 - 7t + 2 = 0$

$(3t-1)(t-2) = 0$

✓

$\tan\theta/2 = 1/3$ or 2

$\theta/2 = 18^\circ 26'$ or $63^\circ 26'$

$\therefore \theta = 36^\circ 52'$,, $126^\circ 52'$

(d) $$\int_0^{\pi/9} \sec 3x \tan 3x \, dx = \frac{1}{3}\left[\sec 3x\right]_0^{\pi/9}$$

$$= \frac{1}{3}\left[\sec \pi/3 - \sec 0\right]$$

$$= \frac{1}{3}(2 - 1)$$

$$= \frac{1}{3}$$

QUESTION 5

(a) Eqn of chord PQ: $m = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$$= \frac{a(p-q)(p+q)}{2a(p-q)}$$

$$= \frac{p+q}{2}$$

$$y - ap^2 = \frac{p+q}{2}(x - 2ap)$$

Since PQ is a focal chord, it passes through the focus $(0, a)$

$$a - ap^2 = \frac{p+q}{2}(0 - 2ap)$$

$$a - ap^2 = -ap(p+q)$$

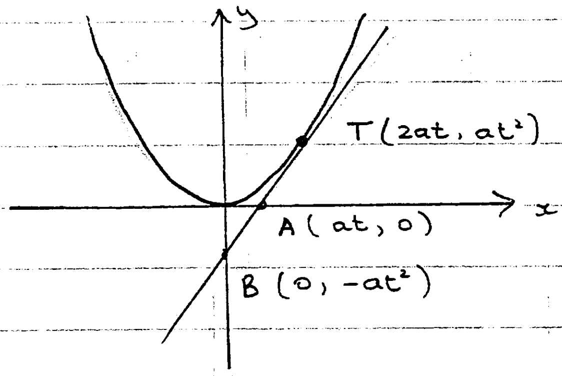
$$a - \cancel{ap^2} = -\cancel{ap^2} - apq$$

$$a = -apq$$

$$\therefore pq = -1$$

(b) i) $y = x^2/4a$

$\frac{dy}{dx} = x/2a$



at T, $m = 2at/2a$

$= t$

$y - at^2 = t(x - 2at)$

$y - at^2 = tx - 2at^2$

$\therefore\ y - tx + at^2 = 0$, is the eqⁿ of the tangent at T.

ii) when $y = 0$, $x = at$ $\therefore$ A is the point $(at, 0)$

when $x = 0$, $y = -at^2$ $\therefore$ B is the point $(0, -at^2)$

iii) Show that ΔSTB is isosceles

$SB = a + at^2$

$= a(1 + t^2)$

$ST^2 = (2at - 0)^2 + (at^2 - a)^2$

$= 4a^2t^2 + a^2t^4 - 2a^2t^2 + a^2$

$= a^2t^4 + 2a^2t^2 + a^2$

$= a^2(t^4 + 2t^2 + 1)$

$= a^2(t^2 + 1)^2$

$ST = a(t^2 + 1)$

$\therefore\ SB = ST$

ie/ the tangent is equally inclined to the y-axis and the focal chord through T.

(iv) $A(\overset{x_1}{at}, \overset{y_1}{0})$ $B(\overset{x_2}{0}, \overset{y_2}{-at^2})$ $k : l$ $T(\overset{x}{2at}, \overset{y}{at^2})$

$2at = \frac{k.0 + lat}{k+l}$ $at^2 = \frac{-kat^2 + l.0}{k+l}$ * or, use the diag.

$k + l = \frac{lat}{2at}$ $k + l = \frac{-kat^2}{at^2}$

$= \frac{l}{2}$ $= -k$

$\therefore\ \frac{l}{2} = -k$

ie/ $\frac{k}{l} = -\frac{1}{2}$

$\therefore$ T divides AB externally in the ratio 1 : 2

QUESTION 6.

a) $\log_3\left(\dfrac{9x-2}{x^2}\right) = 2$

$\therefore \dfrac{9x-2}{x^2} = 3^2$

$9x - 2 = 9x^2$

$9x^2 - 9x + 2 = 0$

$(3x-2)(3x-1) = 0$

$\therefore x = 2/3, 1/3$

(b) $\displaystyle\int_0^1 \frac{3x^2}{1+x^3}\,dx = \Big[\log(1+x^3)\Big]_0^1$

$= \log 2 - \log 1$

$= \log 2$

(c) $V = 2\pi \displaystyle\int_0^1 (e^{2x} + 2 + e^{-2x})\,dx$

$y = e^{x} + e^{-x}$

$y^2 = e^{2x} + 2 + e^{-2x}$

$= 2\pi \left[\dfrac{e^{2x}}{2} + 2x - \dfrac{e^{-2x}}{2}\right]_0^1$

$= 2\pi\left[\dfrac{e^2}{2} + 2 - \dfrac{e^{-2}}{2} - \left(\dfrac{1}{2} - \dfrac{1}{2}\right)\right]$

$\therefore$ vol. is $4\pi\left(e^2 + 4 - 1/e^2\right)$ cub. units.

(d) $y = xe^{-x}$

$u = x \qquad v = e^{-x}$

$u' = 1 \qquad v' = -e^{-x}$

$\dfrac{dy}{dx} = e^{-x} - xe^{-x}$

$\dfrac{d^2y}{dx^2} = -e^{-x} - \left(e^{-x} - xe^{-x}\right)$

$= -e^{-x} - e^{-x} + xe^{-x}$

$= e^{-x}(x-2)$

St. pts. occur when $y' = 0$

ie/ $e^{-x}(1-x) = 0$

$\therefore x = 1$

when $x = 1$, $y = e^{-1}$, $y'' < 0$

$\therefore$ there is a max. turning point at $(1, 1/e)$

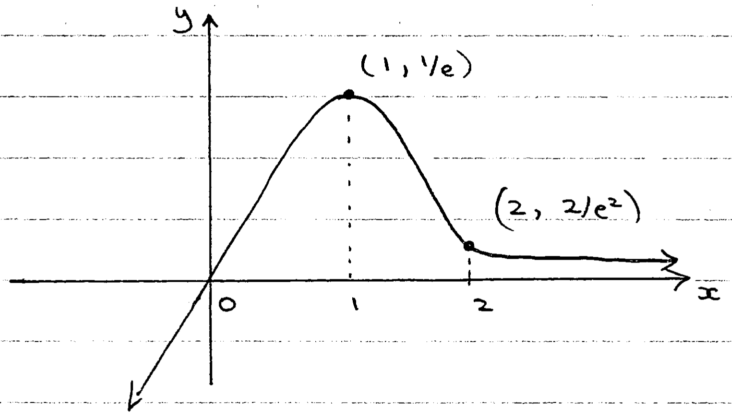
ii) Inflexion occurs when $y'' = 0$ and concavity changes.

$y'' = 0$ when $x = 2$, $y = 2e^{-2}$

$$\left.\begin{array}{l} \text{when } x = 1,\ y'' < 0 \\ \quad " \quad x = 3,\ y'' > 0 \end{array}\right\} \quad \therefore \text{ concavity has changed.}$$

$(2, 2/e^2)$ is a point of inflexion

iii)



QUESTION 7.

a) $$\int_0^4 x(16-x^2)^{\frac{1}{2}}\,dx = -\frac{1}{2}\int_0^4 -2x(16-x^2)^{1/2}\,dx$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\left[(16-x^2)^{3/2}\right]_0^4$$

$$= -\frac{1}{3}\left[0 - 16^{3/2}\right]$$

$$= 64/3$$

OR// $$\int_0^4 (16-x^2)^{1/2}\cdot x\,dx = -\frac{1}{2}\int_{16}^{0} u^{1/2}\cdot du$$

$u = 16 - x^2$

$du = -2x\,dx$

$x = 0$, $u = 16$

$x = 4$, $u = 0$

$$= -\frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_{16}^{0}$$

$$= -\frac{1}{3}\left[0 - 16^{3/2}\right]$$

$$= 64/3$$

b) $V = \pi \int_0^{\pi/3} \tan^2 x \, dx$ $\qquad \sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$= \pi \int_0^{\pi/3} (\sec^2 x - 1) \, dx$

$= \pi \left[\tan x - x \right]_0^{\pi/3}$

$= \pi \left[\tan \pi/3 - \pi/3 - (\tan 0 - 0) \right]$

$= \pi \left(\sqrt{3} - \pi/3 \right)$

$\therefore$ volume is $\pi/3 (3\sqrt{3} - \pi)$ cub. units.

c) $\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

ie/ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\int_0^{\pi/2} \sin^2 \frac{\theta}{2} \, d\theta = \frac{1}{2} \int_0^{\pi/2} (1 - \cos \theta) \, d\theta$

$= \frac{1}{2} \left[\theta - \sin \theta \right]_0^{\pi/2}$

$= \frac{1}{2} \left[\frac{\pi}{2} - \sin \frac{\pi}{2} - (0 - \sin 0) \right]$

$= \frac{1}{2} \left[\frac{\pi}{2} - 1 \right]$

$= \frac{1}{4} (\pi - 2)$

d) $f''(x) = -18 \cos 3x$

$f'(x) = -6 \sin 3x + c$ $\qquad f'(x) = 0$ when $x = \frac{2\pi}{3}$

$0 = -6 \sin 2\pi + c$ $\qquad$ ie/ $c = 0$

$f(x) = 2 \cos 3x + k$

$1 = 2 \cos 2\pi + k$ $\qquad$ ie/ $k = -1$

$\therefore f(x) = 2\cos 3x - 1$