SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

APRIL 2004

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 2

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in 3 sections. Section A (Questions 1 - 3), Section B (Questions 4 - 6) and Section C (Questions 7 - 8).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 87 Marks

- Attempt questions 1- 8
- All questions are NOT of equal value.

Examiner: *R. Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 87
Attempt Questions 1 – 8
All questions are NOT of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (11 marks) **Marks**

(a) Find $\int e^{\frac{x}{2}}\,dx$ **1**

(b) Find the largest possible (natural) domain of the following function, **2**

$$y = \log_e\left(\sin^{-1} x\right)$$

(c) Find a primitive function for

(i) $\dfrac{3x}{4+x^2}$ **1**

(ii) $\dfrac{3}{4+x^2}$ **2**

(d) Differentiate $y = \log_e\left(\sin^{-1} x\right)$ **2**

(e) Solve $\tan\theta = \sin 2\theta$ for $0 < \theta < \pi$ **3**

Section A (continued)

Question 2 (13 marks) **Marks**

(a) Show that $\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ **2**

(b) (i) Show that $\dfrac{d}{d\theta}\left(\tan^3\theta\right) = 3\sec^2\theta(\sec^2\theta - 1)$ **2**

(ii) Hence, or otherwise, evaluate **3**

$$\int_0^{\frac{\pi}{4}} \sec^4\theta \, d\theta$$

(c) Show that $\dfrac{d}{dx}\left(\dfrac{\tan x}{e^{2x}}\right) = \left(\dfrac{\tan x - 1}{e^x}\right)^2$ **3**

(d) Find the possible values of m if $y = e^{-mx}$ satisfies **3**

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

Question 3 (12 marks)

(a) Find the exact value of **3**

$$\int_{\frac{3\sqrt{3}}{2}}^{3} \frac{2}{\sqrt{9-x^2}}\, dx$$

(b) Write down the general solution, in terms of π, of **4**

$$2\sin\theta = \tan\theta$$

(c) Solve the equation **3**

$$2\ln(3x+1) - \ln(x+1) = \ln(7x+4)$$

(d) Find $\displaystyle\int \frac{x+1}{x^2+2x-5}\, dx$ **2**

SECTION B (Use a SEPARATE writing booklet)

Question 4 (10 marks) **Marks**

(a) State the domain of $y = 2\sin^{-1} x$ **1**

(b) Prove that $\frac{d}{dx}\left(3^x\right) = 3^x.\ln 3$ **2**

(c) Show that $\log_4 9 + \log_4 8 - 2\log_4 6 = \frac{1}{2}$ **2**

(d) If $f(x) = x^2 + 2$ and $g(x) = 2x + 3$, find $f(g(x))$ **2**

(e) Differentiate with respect to x **3**

$$y = \ln\left(\frac{2x(x-1)^3}{\sqrt{x+1}}\right)$$

[Hint: Do not combine the answer as a single fraction]

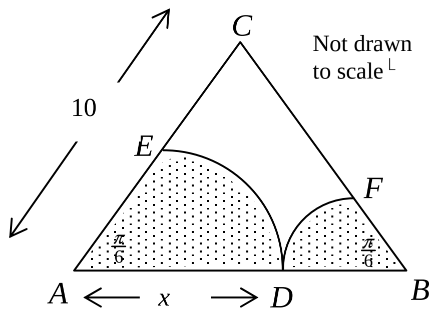
Question 5 (13 marks)

(a) Evaluate $\int_0^{\frac{\pi}{4}} \sec^2 x.e^{\tan x}\, dx$ using the substitution $u = \tan x$. **3**

Leave your answer in *exact* form.

(b) Given $y = e^{\sin x}$, solve $\frac{d^2y}{dx^2} - y = 0$ for $0 \le x \le 2\pi$ **4**

(c)



ΔABC is isosceles with $AC = BC = 10$, $\angle ABC = \angle CAB = \pi/6$, $AD = x$. *AED* and *BDF* are sectors of circles with radii *AD* and *DB* respectively.

(i) Find an expression for *BD*. **3**

(ii) Show that the sum of the areas of the sectors *AED* and *BDF* is given by $\frac{\pi}{12}\left(2x^2 - 20\sqrt{3}x + 300\right)$ cm^2 **3**

Question 6 (12 marks) **Marks**

(a) (i) Find $\int \cos^2 \theta d\theta$ **2**

(ii) Hence, find $\int \frac{dx}{(1+x^2)^2}$ using the substitution $x = \tan\theta$ **4**

(b) (i) Sketch the graph of the function $f(x) = e^x - 4$, showing clearly the coordinates of any points of intersection with the coordinate axes and the equations of any asymptotes. **2**

(ii) On the same diagram, sketch the graph of the inverse function, $y = f^{-1}(x)$, showing clearly the coordinates of any points of intersection with the coordinate axes and the equations of any asymptotes. **2**

(iii) Show that the *x* coordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^x - x - 4 = 0$. **2**

SECTION C (Use a SEPARATE writing booklet)

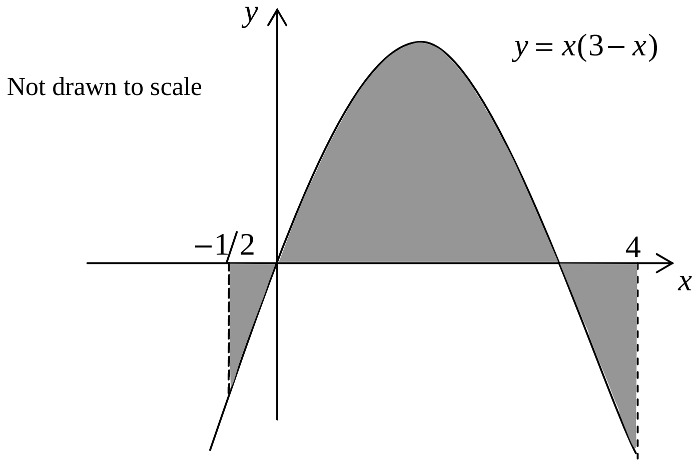
Question 7 (7 marks) **Marks**

(a) The region bounded by $y = \log_e x$, $x = 2$, $x = 5$ and the x axis is rotated about the x axis. **3**

Use the Trapezoidal Rule with four function values to find an approximation to this volume.

Express your answer correct to 2 decimal places.

(b) **4**

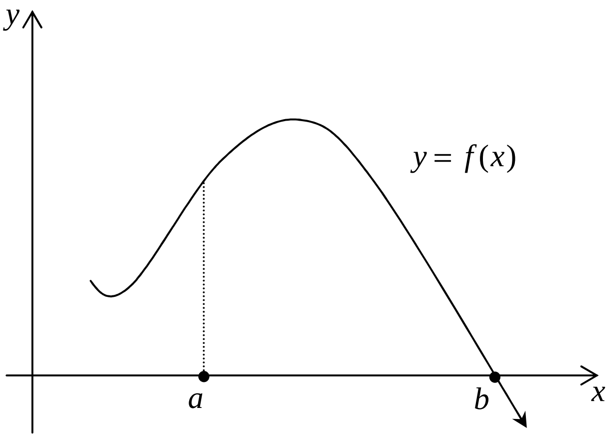


Find the size of the shaded area, correct to 2 decimal places.

SECTION C (continued)

Question 8 (9 marks) **Marks**

(a)



Consider the above graph of $y = f(x)$. The value $x = a$ shown on the x axis is taken as the first approximation to the solution $x = b$ of $f(x) = 0$. **3**

Is the second approximation obtained by Newton's Method a better approximation to b than the first approximation?

Justify your answer (using the diagram in your answer).

(b) Consider $\tan^{-1} y = 2\tan^{-1} x$

(i) Express y as a function of x, independent of any trigonometric ratio. **2**

(ii) Show that the function has no turning points. **3**

(iii) State the domain of the function. **1**

THIS IS THE END OF THE PAPER

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

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ASSESSMENT TASK # 2

Mathematics Extension 1
Sample Solutions

Mathematics Extension 1
Assessment 2
Year 12 2004.

Question 1

(a) $2e^{\frac{\pi}{2}} + C$

(b) $\{x : 0 < x \leq 1\}$

(c) (i) $\frac{3}{2}\ln(4+x^2)$

(ii) $\frac{3}{2}\tan^{-1}\frac{x}{2}$

(d) $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}\sin^{-1}x}$

(e)

$$\tan\theta = \sin 2\theta \quad 0<\theta<\pi$$

$$\tan\theta - \sin 2\theta = 0$$

$$\frac{\sin\theta}{\cos\theta} - 2\sin\theta\cos\theta = 0$$

$$\sin\theta(\sec\theta - 2\cos\theta) = 0$$

$$\therefore \sin\theta = 0$$

$$\theta = 0, \pi$$

However $0<\theta<\pi$.

$$\therefore \sec\theta - 2\cos\theta = 0$$

$$1\text{-}2\cos^2\theta = 0$$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos\theta = \pm\frac{1}{\sqrt{2}}$$

$$\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

Question 2

(a) $\tan 75^0 = \tan(30^0 + 45^0)$

$$= \frac{\tan 30^0 + \tan 45^0}{1 - \tan 30^0 \tan 45^0}$$

$$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}.1}$$

$$= \frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

(b)(i) Let $u = \tan\theta$

$$\frac{du}{d\theta} = \sec^2\theta$$

$$y = u^3$$

$$\frac{dy}{du} = 3u^2$$

$$\frac{dy}{dx} = \frac{du}{d\theta}.\frac{dy}{du}$$

$$= \sec^2\theta.3u^2$$

$$= 3\sec^2\theta.\tan^2\theta$$

$$= 3\sec^2\theta(\sec^2\theta - 1)$$

$$\therefore \frac{d}{d\theta}\tan^3\theta = 3\sec^2\theta(\sec^2\theta - 1)$$

(ii) $\frac{d}{d\theta}\tan^3\theta = 3\sec^2\theta(\sec^2\theta - 1)$

$$\therefore \int 3\sec^4\theta - 3\sec^2\theta \, d\theta = \tan^3\theta + C$$

$$\int 3\sec^4\theta \, d\theta - \int 3\sec^2\theta \, d\theta = \tan^3\theta + C$$

$$\int \sec^4\theta \, d\theta - \int \sec^2\theta \, d\theta = \frac{1}{3}\tan^3\theta + C$$

$$\int \sec^4\theta \, d\theta = \int \sec^2\theta \, d\theta + \frac{1}{3}\tan^3\theta + C$$

$$= \tan\theta + \frac{1}{3}\tan^3\theta + C$$

$$\therefore \int_0^{\frac{\pi}{4}} \sec^4\theta \, d\theta = \left[\tan\theta + \frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{4}}$$

$$= (\tan\frac{\pi}{4} + \frac{1}{3}\tan^2\frac{\pi}{4}) - (\tan 1 + \frac{1}{3}\tan^2 1)$$

$$= \frac{1}{3} + 1$$

$$= 1\frac{1}{3}$$

(c) Let $u = \tan x$

$$\frac{du}{dx} = \sec^2 x$$

$$v = e^{2x}$$

$$\frac{dv}{dx} = 2e^{2x}$$

$$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

$$= \frac{(e^{2x})(\sec^2 x) - (\tan x)(2e^{2x})}{(e^{2x})^2}$$

$$= \frac{\sec^2 x - 2\tan x}{e^{2x}}$$

$$= \frac{(\tan^2 x + 1) - 2\tan x}{e^{2x}}$$

$$= \frac{\tan^2 x - 2\tan x + 1}{(e^x)^2}$$

$$= \frac{(\tan x - 1)^2}{(e^x)^2}$$

$$= \left(\frac{\tan x - 1}{e^x}\right)^2$$

(d)

$$y = e^{-mx}$$

$$\frac{dy}{dx} = -me^{-mx}$$

$$\frac{d^2y}{dx^2} = m^2 e^{-mx}$$

$$m^2 e^{-mx} + 2(-me^{-mx}) - 15e^{-mx} = 0$$

$$e^{-mx}(m^2 - 2m - 15) = 0$$

$\therefore e^{-mx} = 0$

no solution

$\therefore m^2 - 2m - 15 = 0$

$(m-5)(m+3) = 0$

$m = 5, -3$

Question 3

(a) $$\int_{\frac{3\sqrt{3}}{2}}^{3} \frac{2}{\sqrt{9-x^2}}dx = 2\int_{\frac{3\sqrt{3}}{2}}^{3} \frac{1}{\sqrt{9-x^2}}dx$$

$$= 2\int_{\frac{3\sqrt{3}}{2}}^{3} \frac{1}{\sqrt{3^2-x^2}}dx$$

$$= 2\left[\sin^{-1}\frac{x}{3}\right]_{\frac{3\sqrt{3}}{2}}^{3}$$

$$= 2\left(\sin^{-1}\frac{3}{3} - \sin^{-1}\frac{3\sqrt{3}}{6}\right)$$

$$= 2\left(\sin^{-1}1 - \sin^{-1}\frac{\sqrt{3}}{2}\right)$$

$$= 2\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$= \frac{\pi}{3}$$

(b) $$2\sin\theta = \tan\theta$$

$$2\sin\theta = \frac{\sin\theta}{\cos\theta}$$

$$2\sin\theta\cos\theta - \sin\theta = 0$$

$$\sin\theta(2\cos\theta - 1) = 0$$

$$\therefore \sin\theta = 0$$

$$\theta = \pi n + (-1)^n \sin^{-1}(0), \quad n \in Z$$

$$= \pi n + (-1)^n(0)$$

$$= \pi n$$

$$\therefore 2\cos\theta - 1 = 0$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = 2\pi n \pm \cos^{-1}\left(\frac{1}{2}\right), \quad n \in Z$$

$$= 2\pi n \pm \frac{\pi}{3}$$

(c) $2\ln(3x+1)-\ln(x+1)=\ln(7x+4)$

$$\frac{(3x+1)^2}{x+1}=7x+4$$

$$9x^2+6x+1=(x+1)(7x+4)$$

$$9x^2+6x+1=7x^2+11x+4$$

$$2x^2-5x-3=0$$

$$(2x+1)(x-3)=0$$

$\therefore 2x+1=0$ or $x-3=0$

$x=-\frac{1}{2}$ $x=3$

Since $x>0$ a log of a negative is undefined, therefore x=3 is the only solution.

(d) $\frac{1}{2}\ln(x^2+2x-5)+C$

Q4

(a) $-1 \le x \le 1$ (1)

b) $y = 3^x$

$x = \log_3 y$

$= \frac{\log_e y}{\log_e 3}$

$\frac{dx}{dy} = \frac{1}{\ln 3y}$

$\frac{dy}{dx} = \frac{d3^x}{dx} = 3^x \ln 3$ (2)

(c) $x = \log_4\left(\frac{9 \times 8}{6^2}\right)$

$x = \log_4 2$

$4^x = 2$

$x = \frac{1}{2}$ (2)

(d) $f[g(x)] = (2x+3)^2 + 2$

$= 4x^2 + 12x + 11$ (2)

(e) $y = \ln 2x + 3\ln(x-1) - \frac{1}{2}\ln(x+1)$

$y' = \frac{1}{x} + \frac{3}{x-1} - \frac{1}{2(x-1)}$ (3)

Q5 (a) $\int_0^{\pi/4} \sec^2 x \, e^{\tan x} dx$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$= \int_0^1 \sec^2 x \, e^u \frac{du}{\sec^2 x}$

$x = \frac{\pi}{4} \quad u = 1$

$x = 0 \quad u = 0$

$= [e^u]_0^1$

$= e - 1$ (3)

b) $y = e^{\sin x}$

$y' = e^{\sin x} \cos x$

$y'' = \cos x \, e^{\sin x} \cos x + e^{\sin x} . -\sin x$

$y'' - y = \cos^2 x \, e^{\sin x} - \sin x \, e^{\sin x} - e^{\sin x}$

$e^{\sin x}(\cos^2 x - \sin x - 1) = 0$

$1 - \sin^2 x - \sin x - 1 = 0 \quad (e^{\sin x} \neq 0)$

$-\sin x(\sin x + 1) = 0$

$\sin x = 0$ or $\sin x = -1$

$x = 0, \pi, 2\pi$ or $3\pi/2$ (4)

Q5(c)

(i) In $\triangle ABC$ $\angle ACB = \frac{2\pi}{3}$

$AB^2 = 10^2 + 10^2 - 2 \times 100 \cos \frac{2\pi}{3}$ (Cos Rule

$AB = 10\sqrt{3}$

$DB = 10\sqrt{3} - x$ (3)

(ii) Areas$(AED + BDF) = \frac{x^2 \pi}{2 \cdot 6} + \frac{1}{2}(10\sqrt{3} - x)^2 \times \frac{\pi}{6}$

$= \frac{\pi}{12}(2x^2 + 300 - 20\sqrt{3}x)$ (3)

Q6(a)(i) $\cos 2\theta = 2\cos^2\theta - 1$

$\int \cos^2\theta \, d\theta = \frac{1}{2}\int(\cos 2\theta + 1) \, d\theta$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + C$ (2)

(ii) $\int \frac{dx}{(1+x^2)^2}$

$x = \tan\theta$

$\frac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta \, d\theta$

$= \int \frac{\sec^2\theta}{(1+\tan^2\theta)^2} d\theta$

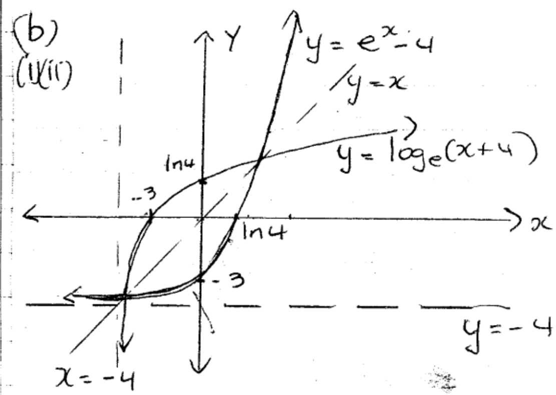
$1 + \tan^2\theta = \sec^2\theta$

$= \int \frac{\sec^2\theta}{(\sec^2\theta)^2} d\theta$

$= \int \frac{1}{\sec^2\theta} d\theta = \int \cos^2\theta \, d\theta$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + C$ (from (i)

(b) (i)(ii)



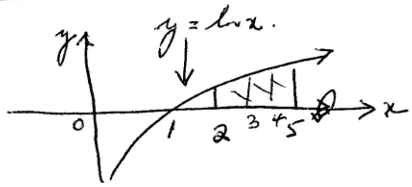
(iii) The curves meet on the line $y = x$.

$e^x - 4 = x$

$e^x - x - 4 = 0$

QUESTION 7

(a)



$V \doteq \pi \int_2^5 (\ln x)^2 \, dx$ ✓

$= \pi \cdot \frac{1}{2} \Big[(\ln 2)^2 + (\ln 5)^2 + 2\left[(\ln 3)^2 + (\ln 4)^2 \right] \Big]$ ✓

$= 14.065 \, u^3$ (3.D.P.) ✓

(b) Shaded area $= \left| \int_{-\frac{1}{2}}^{0} x(3-x)\,dx \right| + \int_0^3 x(3-x)\,dx + \left| \int_3^4 x(3-x)\,dx \right|$

$= \left| \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_{-\frac{1}{2}}^{0} \right| + \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 + \left| \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_3^4 \right|$

$= \left| 0 - \left(\frac{3/4}{2} + \frac{1/8}{3} \right) \right| + \left(\frac{3 \times 9}{2} - \frac{27}{3} \right) + \left| \left(24 - \frac{64}{3} \right) - \left(\frac{3 \times 9}{2} - \frac{27}{3} \right) \right|$

✓ ✓ ✓

$= \frac{5}{12} + \frac{9}{2} + \frac{11}{6}$

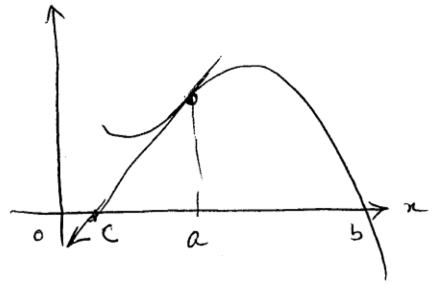
$= \frac{5 + 54 + 22}{12}$

$= \frac{81}{12}$ ✓

$= \frac{27}{4} \, u^2$ or $6\frac{3}{4} \, u^2 = 6.75 \, u^2$

QUESTION 8.

(a)



NO. The tangent to $y = f(x)$ where $x = a$ will cross the x-axis further from b than a, caused by the turning point between the root and the 1st approximation.

(3) ✓✓✓

must use a diagram

(b) (i) $\tan^{-1} y = 2\tan^{-1} x$.

$\therefore \tan(\tan^{-1} y) = \tan(2\tan^{-1} x)$ ' to this angle.

$y = \dfrac{2\tan(\tan^{-1} x)}{1 - \tan^2(\tan^{-1} x)}$

$\therefore \boxed{y = \dfrac{2x}{1-x^2}}$ ✓✓ (2)

(ii) now $\dfrac{dy}{dx} = \dfrac{(1-x^2).2 - 2x.-2x.}{(1-x^2)^2}$

$= \dfrac{2 - 2x^2 + 4x^2}{(1-x^2)^2}$

$= \dfrac{2(1+x^2)}{(1-x^2)^2}$ ← must get this (3)

Clearly for $\dfrac{dy}{dx} = 0$, $1 + x^2 = 0$ which has no real solutions.

$\therefore \dfrac{dy}{dx} \neq 0$ $\therefore$ no turning points ✓✓✓

(iii) DOMAIN: $\boxed{\text{All real} \neq \pm 1.}$ ✓ (1)