SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2007
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #2

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 61

- Attempt questions 1 - 3

Examiner: *A Fuller*

Total marks – 61
Attempt all questions
All questions are NOT of equal value

Answer each SECTION in a SEPARATE writing booklet

Marks

Section A

Question 1 (19 marks)

(a) Differentiate $\sin^{-1} 3x$ **1**

(b) Find the exact value of the following:

(I) $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ **1**

(II) $\tan^{-1}(-1)$ **1**

(III) $\sin^{-1}\left(\cos^2 \dfrac{\pi}{4}\right)$ **1**

(c) For the function $f(x) = 2\cos^{-1}\left(\dfrac{x}{2}\right)$

(I) Write down the domain and range **2**

(II) Sketch the graph of $y = f(x)$ **1**

(III) Find $f'(x)$ **1**

(d) Find the indefinite integral of $\dfrac{1}{\sqrt{1-4x^2}}$ **2**

(e) ABCD is a cyclic quadrilateral in which the opposite sides AB and DC are equal.

(I) Draw a diagram

(II) Prove that the diagonals AC and BD are equal **3**

(f) Find the value of k if $\displaystyle\int_0^1 \frac{dx}{x^2+3} = k\pi$ **2**

(g) A committee of 4 is to be formed from a group of 10 people containing 6 men and 4 women

(I) In how many ways can this committee be formed so that it always contains at least one woman? **2**

(II) If the selection is determined by the drawing of names from a hat, what is the probability that the committee will contain a majority of women? **2**

End of Section A

Section B (Use a SEPARATE writing booklet)

Question 2 (21 marks)

(a) Use the substitution $u = x^2 - 4$ to find an expression for $\int \frac{2x}{\sqrt{x^2-4}}dx$ **2**

(b) An urn contains 3 red, 4 white and 5 blue marbles. Three marbles are drawn at random in succession, and after each marble is drawn it is replaced in the urn. What is the probability that:

(I) marbles of different colours are drawn? **1**

(II) red, white, and blue marbles are drawn in that order? **1**

(III) at least 2 red marbles are drawn? **1**

(c) Use the method of mathematical induction to prove that **3**

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\ldots\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

(d) Find the exact area between the curve $y=\sin^{-1}x$, the x-axis, and the lines $x=0$ and $x=\frac{1}{2}$ **2**

(e) Find $\cos\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$ **2**

(f) Use Newton's method once to find an approximation for $\sqrt{4.02}$ which is better than 2 **2**

(g) (I) How many five digit numbers can be formed from the digits 1, 3, 5, 7, 9

(α) without repeating any digit? **1**

(β) allowing each digit to be used any number of times? **1**

(II) Consider now the numbers less than 5000 which can be formed from four or fewer of the digits 1, 3, 5, 7, 9 (without repeats)

(α) How many of these are there? **1**

(β) One of these numbers is selected at random. What is the probability that it is divisible by 5? **1**

(h) Prove by induction that $2^{3n} - 3^{n}$ is divisible by 5, if n is a positive integer **3**

End of Section B

Section C (Use a SEPARATE writing booklet)

Question 3 (21 marks)

(a) Find the volume formed when the area between the curve $y = \sin x \cos x$, the x-axis, and the lines $x = 0$ and $x = \frac{\pi}{3}$ is rotated about the x-axis **3**

(b) Find the value of $\int_0^{\frac{\pi}{2}} \frac{\sec^2(\sin x)}{\sec x} dx$ using the substitution $u = \sin x$ **2**

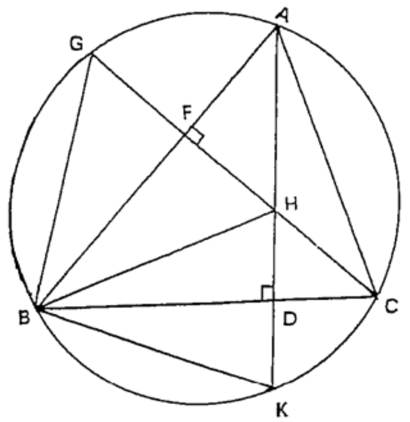
(c) Given that $y = \sin^{-1}\sqrt{x}$ show that $\frac{dy}{dx} = \frac{1}{\sin 2y}$ **2**

(d) State the largest domain and the corresponding range of the function $y = \sin^{-1}\left(\sqrt{1-x^2}\right)$ **2**

(e) (I) Find $\frac{dy}{dx}$ for $y = \cos^{-1}(\sin x)$ **2**

(II) Hence, or otherwise, sketch $y = \cos^{-1}(\sin x)$ for $-2\pi \le x \le 2\pi$ **2**

(f)



In the above diagram, ABC is a triangle inscribed in a circle.
The perpendicular from A onto BC meets it at D and is then produced to meet the circumference at K.
The perpendicular from C onto AB meets it at F and is then produced to meet the circumference at G.
The two perpendiculars AD and CF meet at the point H.

(I) Show that the quadrilaterals AFDC and BFHD are both cyclic **1**

(II) Prove that AB bisects the angle GBH **2**

(III) Prove that GB = BK **2**

(g) Given that $\sin x + \cos x = 2^{\frac{1}{2}} \sin\left(x + \frac{\pi}{4}\right)$ and $y = e^x \sin x$, prove by mathematical induction, that $\frac{d^n y}{dx^n} = 2^{\frac{n}{2}} e^x \sin\left(x + \frac{n\pi}{4}\right)$ **3**

where $\frac{d^n y}{dx^n}$ denotes the n^{th} derivative of y with respect to x

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

3 unit Section A 2007 May Assess Task 3. (19)

1 (a) $\frac{d}{dx}(\sin^{-1}3x) = \frac{1}{\sqrt{1-9x^2}} \times 3 = \frac{3}{\sqrt{1-9x^2}}$ (1)

(b) (i) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ (30°) (1)

(ii) $\tan^{-1}(-1) = -\tan^{-1}1 = -\frac{\pi}{4}$ (-45°) (1)

(iii) $\sin^{-1}\left(\cos^2\frac{\pi}{4}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (30°) (1)

(c) (i) $f(x) = 2\cos^{-1}\left(\frac{x}{2}\right)$

domain of $f(x) = \cos^{-1}x$; $-1 \le x \le 1$

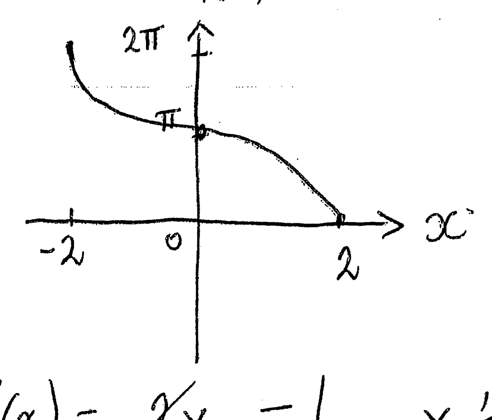
domain of $f(x) = 2\cos^{-1}\left(\frac{x}{2}\right)$: $-1 \le \frac{x}{2} \le 1$

$-2 \le x \le 2$ (1)

range of $f(x) = \cos^{-1}x$: $0 \le y \le \pi$.

range of $f(x) = 2\cos^{-1}\left(\frac{x}{2}\right)$: $0 \le 2\cos^{-1}\left(\frac{x}{2}\right) \le 2\pi$ (1)

(ii)



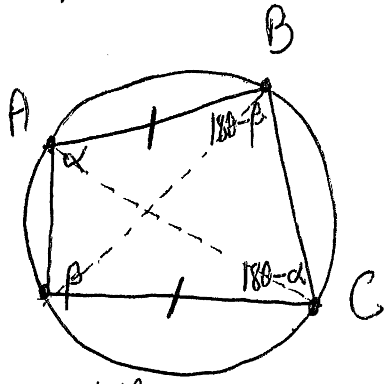
(1)

(iii) $f'(x) = \cancel{2} \times \frac{-1}{\sqrt{1-\frac{x^2}{4}}} \times \frac{1}{\cancel{2}}$

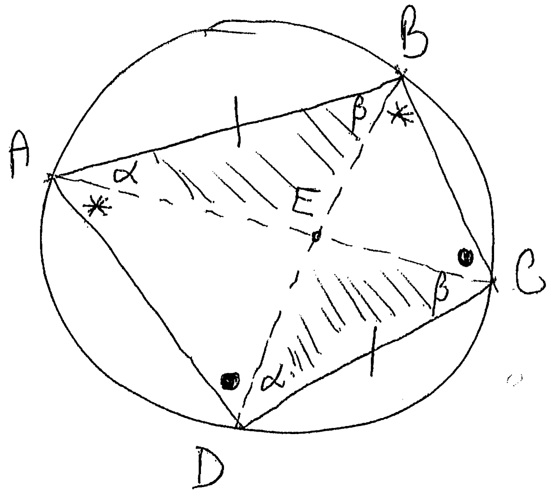
$= \frac{-1}{\sqrt{\frac{4-x^2}{4}}} = \frac{-1}{\frac{\sqrt{4-x^2}}{2}} = \frac{-2}{\sqrt{4-x^2}}$ (1)

(d) $\frac{1}{2}\int \frac{2\,1}{\sqrt{1-4x^2}}\,dx \quad = \quad \frac{1}{2}\sin^{-1}2x + c$ (2)

(e) (i)



$\Rightarrow$



$B\hat{A}E = B\hat{D}C = \alpha$
angles standing on same arc.

$A\hat{B}E = D\hat{C}E = \beta$
angles standing on same arc

$\triangle ABE \equiv \triangle DCE$ (AAS)

$\therefore$ $AE = DE$, $BE = CE$ } corresponding sides in congruent triangles

$AC = AE + CE$
$DB = DE + BE$
$\therefore$ $AC = DB$ as needed. (3)

(f) $\int_0^1 \frac{1\,dx}{x^2+3} = k\pi$

$\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\Big]_0^1 = \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}0\right)$
$= \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6}$

So $k\pi = \frac{\pi}{6\sqrt{3}}$

$k = \frac{1}{6\sqrt{3}}$ or $\frac{1}{6\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{18}$ (2)

(g) 10 people : 6 men, 4 women
Committee of 4.

(i) at least 1 woman

1 woman 3 men + 2 women 2 men + 3 women 1 man + 4 women

$${}^4C_1 \times {}^6C_3 + {}^4C_2 \times {}^6C_2 + {}^4C_3 \times {}^6C_1 + {}^4C_4$$

$$= 4 \times \frac{\cancel{6}\times5\times4}{\cancel{3}\times\cancel{2}\times1} + \frac{\overset{2}{\cancel{4}}\times3}{\cancel{2}\times1} \times \frac{\overset{3}{\cancel{6}}\times5}{\cancel{2}\times1} + \frac{4\times\cancel{3}\times\cancel{2}}{\cancel{3}\times\cancel{2}\times\cancel{1}} \times 6 + 1$$ 2

$$= 80 + 90 + 24 + 1 = 195$$
ways

(ii) sample space is $${}^{10}C_4 = \frac{10\times\overset{3}{\cancel{9}}\times\cancel{8}\times7}{\cancel{4}\times\cancel{3}\times\cancel{2}\times1} = 210$$

majority of women ? 3 women 1 man + 4 women = 25

$$\frac{25}{210} = \frac{5}{42}$$ 2

Q2 (a) $u = x^2 - 4$

$\frac{du}{dx} = 2x$

$du = 2x \cdot dx$.

$\therefore \int \frac{2x\,dx}{\sqrt{x^2-4}} = \int \frac{du}{\sqrt{u}}$

$= \int u^{-\frac{1}{2}}\,du$

$= 2 \cdot u^{\frac{1}{2}} + C$

$= 2\sqrt{x^2-4} + C$

(b) I $P(RWB) = \frac{3}{12} \times \frac{4}{12} \times \frac{5}{12}$

$= \frac{5}{144}$

But R, W & B can be arranged in 6 orders

$\therefore P(\text{3 different colours}) = \frac{5}{144} \times 6$

$= \frac{5}{24}$

II $P(RWB) = \frac{5}{144}$ from I

III $P(E) = P(RRR) + P(RRO) + P(ROR) + P(ORR)$

$= \frac{3}{12} \times \frac{3}{12} \times \frac{3}{12} + 3 \times \left(\frac{3}{12} \times \frac{3}{12} \times \frac{9}{12}\right)$

$= \frac{1}{64} + 3 \times \frac{3}{64}$

$= \frac{10}{64}$

$= \frac{5}{32}$

(c) $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

Step 1: Consider the statement for $n=1$

LHS $= \frac{1}{1\times 3} = \frac{1}{3}$

RHS $= \frac{1}{2\times 1+1} = \frac{1}{3} =$ LHS.

$\therefore$ The statement is true for $n=1$

Step 2: Assume the statement is true for $n=k$.

ie $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$.

Consider the statement for $n=k+1$

ie $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k+1}{2(k+1)+1}$

LHS $= \frac{1}{1\times 3} + \frac{1}{3\times 5} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$ using assumption

$= \frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$= \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$

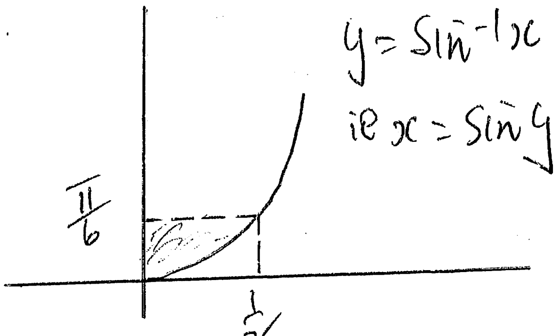
$= \frac{k+1}{2k+3}$

$=$ RHS.

$\therefore$ The statement is true for $n=k+1$ if it is true for $n=k$

Step 3: Now the statement is true for $n = k+1$ if it is true for $n = k$
But it is true for $n = 1$
$\therefore$ It is true for $n = 1+1 = 2$
Then if true for $n = 2$, true for $n = 2+1 = 3$
and so on for all integral n.

(d)



Required area = Area rectangle $- \int_0^{\frac{\pi}{6}} \sin y \, dy$

$= \frac{1}{2} \times \frac{\pi}{6} - \left[-\cos y\right]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{12} - \left[-\frac{\sqrt{3}}{2} - (-1)\right]$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$

(e) Let $\cos\left[\sin^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{\sqrt{3}}{2}\right] = \cos\left[A+B\right]$

then $\sin^{-1}\frac{\sqrt{3}}{2} = A$ and $\cos^{-1}\frac{\sqrt{3}}{2} = B$

$\therefore A = \frac{\pi}{3}$ $\quad B = \frac{\pi}{6}$

ie $\cos(A+B) = \cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$

$= \cos\frac{\pi}{2}$

$= 0.$

(f) $\sqrt{4.02}$ is given by the x intercept of $y = x^2 - 4.02$

ie $f(x) = x^2 - 4.02$ and $f'(x) = 2x$

Now if $x_1 = 2$ is an approximation to a root a better approximation is given by $x_2 = 2 - \frac{f(2)}{f'(2)}$

$= 2 - \frac{4 - 4.02}{4}$

$= 2 - \frac{-0.02}{4}$

$= 2.005$

(g) I (α) $n(S) = 5! = 120$

(β) $n(S) = 5^5 = 3125$

II (α)

4 digits	2 | 4 | 3 | 2	= 48
3 digits	5 | 4 | 3	= 60
2 digits	5 | 4	= 20
1 digit	5	= 5
Total		= 133

(β)

4 digits	2 | 3 | 2 | 1	= 12
3 digits	4 | 3 | 1	= 12
2 digits	4 | 1	= 4
1 digit	1	= 1
		29

$\therefore P(E) = \frac{29}{133}$

(h) Step 1: Consider $n = 1$

$2^3 - 3^1 = 8 - 3 = 5$

$\therefore$ True for $n = 1$

Step 2: Assume $2^{3k} - 3^k = 5A$

Consider $n = k+1$

$2^{3k+3} - 3^{k+1} = 2^{3k}.2^3 - 3^k.3^1$

$= 8.2^{3k} - 3.3^k$

$= 5.2^{3k} + 3.2^{3k} - 3.3^k$

$= 5.2^{3k} + 3(2^{3k} - 3^k)$

$= 5.2^{3k} + 3.5A$

$= 5(2^{3k} + 3A)$

$\therefore$ Statement is true for $n = k+1$ if it is true for $n = k$

(etc)

SECTION C

QUESTION 3 (3)

(a) $V = \pi \int_0^{\frac{\pi}{3}} (\sin x . \cos x)^2 \, dx$ ½

$= \pi \int_0^{\frac{\pi}{3}} \left(\frac{1}{2}\sin 2x\right)^2 dx$ ½

$= \frac{\pi}{4} \int_0^{\frac{\pi}{3}} \sin^2 2x \, dx$

$= \frac{\pi}{4} \int_0^{\frac{\pi}{3}} \frac{1}{2}(1 - \cos 4x) \, dx$ ½

$= \frac{\pi}{8} \left[x - \frac{1}{4}\sin 4x \right]_0^{\frac{\pi}{3}}$ ½

$= \frac{\pi}{8} \left[\left(\frac{\pi}{3} - \frac{1}{4}\sin \frac{4\pi}{3}\right) - 0 \right]$

$= \frac{\pi}{8} \left[\frac{\pi}{3} + \frac{\sqrt{3}}{8} \right]$ ½

(b) $\int_0^{\frac{\pi}{2}} \frac{\sec^2(\sin x)}{\sec x} dx$ (2) let $u = \sin x$

$du = \cos x \, dx$

$= \int_0^{\frac{\pi}{2}} \sec^2(\sin x) \cos x \, dx$ $x = 0, u = 0$; $x = \frac{\pi}{2}, u = 1$

$= \int_0^1 \sec^2 u \, du$ (1)

$= \left[\tan u \right]_0^1 = \tan 1$ (1)

(c) $y = \sin^{-1}\sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$ (1)

$\boxed{x = \sin y}$

(2) $\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{\sin^2 y} \cdot \sqrt{1 - \sin^2 y}}$ (½)

$= \frac{1}{2 \sin y . \cos y}$

$\frac{dy}{dx} = \frac{1}{\sin 2y}$ (½)

(d) (2) $y = \sin^{-1}\left(\sqrt{1-x^2}\right)$

Domain: $-1 \le \sqrt{1-x^2} \le 1$

$\Rightarrow -1 \le x \le 1$

Range: $0 \le y \le \frac{\pi}{2}$

(e)

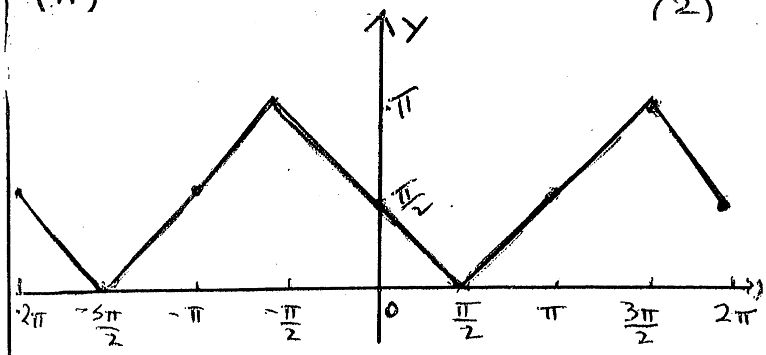
(I) $y = \cos^{-1}(\sin x)$ (2)

$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - \sin^2 x}} . \cos x$

$= \frac{-\cos x}{\sqrt{\cos^2 x}} = \frac{-\cos x}{|\cos x|}$

$\left\{ \begin{array}{l} \text{If } \cos x < 0, \ \frac{dy}{dx} = \frac{-\cos x}{-\cos x} = 1 \\ \text{If } \cos x > 0, \ \frac{dy}{dx} = \frac{-\cos x}{\cos x} = -1 \end{array} \right\}$

(II) (2)



(f) $A\hat{F}D = A\hat{D}C = 90°$

Both angles subtended at the circ. by chord (diameter) AC.

$\therefore$ AFDC cyclic

also

$B\hat{F}H = B\hat{D}H = 90°$ (opp. angles suppl.)

$\Rightarrow$ BDHF cyclic

$G\hat{B}A = G\hat{C}A$ (angles at circ. subtended by arc AG)

$G\hat{C}A = F\hat{D}A$ (angles at circ. subtended by chord AF)

$F\hat{D}A = F\hat{B}H$ (angles at circ. sub. by chord FH BDHF cyclic quad.

$\Rightarrow G\hat{B}A = F\hat{B}H$

$\therefore$ AB bisects $G\hat{B}H$

FDCA is cyclic quad^{l}.

chord FD subtends $F\hat{A}D$ and $F\hat{C}D$ at circ.

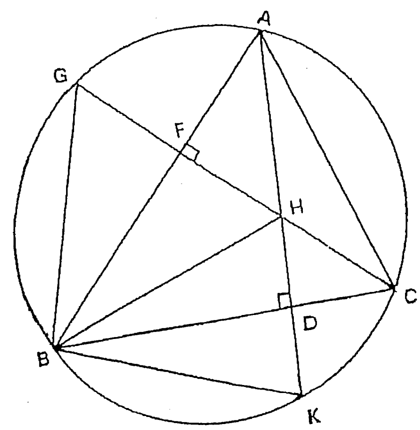
$\therefore F\hat{A}D = F\hat{C}D$

$\Rightarrow$ chord BK subtends $B\hat{A}K = F\hat{A}D$

also chord BG subtends $G\hat{C}B$ at circ. which equals $F\hat{C}D$

$\therefore$ chord BG = chord BK since they subtend equal angles at the cir. of large C.

(*) There are other ways of answering these questions.



(g) $\sin x + \cos x = 2^{\frac{1}{2}} \cdot \sin\left(x + \frac{\pi}{4}\right)$

$y = e^x \sin x$

$\frac{dy}{dx} = e^x [\sin x + \cos x]$

$= e^x . [2^{\frac{1}{2}} \sin(x + \frac{\pi}{4})]$

3) Prove $\frac{d^n y}{dx^n} = 2^{\frac{n}{2}} . e^x \sin\left(x + \frac{n\pi}{4}\right)$ — ✱

When $n = 1$ $\frac{dy}{dx} = 2^{\frac{1}{2}} e^x \sin\left(x + \frac{\pi}{4}\right)$

and $\frac{dy}{dx} = 2^{\frac{1}{2}} e^x \sin\left(x + \frac{1\pi}{4}\right)$ using ✱

$\therefore$ ✱ true for $n = 1$

Assume $\frac{d^k y}{dx^k} = 2^{\frac{k}{2}} e^x \sin\left(x + \frac{k\pi}{4}\right)$

RTP $\frac{d^{k+1}}{dx^{k+1}}(y) = 2^{\frac{k+1}{2}} e^x \sin\left[x + \frac{(k+1)\pi}{4}\right]$

$$\text{LHS} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$$

$$= \frac{d}{dx}\left[2^{k/2} e^x \sin\left(x + \frac{k\pi}{4}\right)\right]$$

by Prod. rule $\Rightarrow$ $$= 2^{k/2} . e^x \left[\cos\left(x + \frac{k\pi}{4}\right) + \sin\left(x + \frac{k\pi}{4}\right)\right]$$

$$= 2^{k/2} e^x \left[2^{\frac{1}{2}} \sin\left(x + \frac{k\pi}{4} + \frac{\pi}{4}\right)\right]$$

$$= 2^{\frac{k+1}{2}} e^x \sin\left[x + \frac{(k+1)\pi}{4}\right]$$

$$= \text{RHS}$$