SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

APRIL 2010
TASK #2
YEAR 12

Mathematics Ext 1

General Instructions:

- Reading time—5 minutes.
- Working time—90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Answer in simplest exact form unless otherwise stated.

Total marks—72 Marks

- Attempt questions 1–6.
- The mark-value of each question is boxed in the right margin.
- Start each NEW section in a separate answer booklet.
- Hand in your answer booklets in 3 sections:
  Section A (Questions 1 and 2),
  Section B (Questions 3 and 4),
  Section C (Questions 5 and 6),

Examiner: Mr D. Hespe

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Section A

Marks

Question 1 (12 points)

(a) (i) How many four-letter arrangements can be made from the letters IOLS? $\boxed{1}$

(ii) In the Herald's *Target* competition, arrangements ending in S are not allowed. How many four-letter *Target* arrangements can be made from the letters IOLS? $\boxed{1}$

(b) Prove that a line from the centre of a circle to the midpoint of a chord is perpendicular to the chord. $\boxed{3}$

(c) Use the graph of $y = \sin x$ to illustrate why $\boxed{2}$

$$\int_{-1}^{1} \sin x \, dx = 0.$$

(d) Evaluate $\lim_{x \to 0} \left\{ \dfrac{\tan 4x}{7x} \right\}$. $\boxed{1}$

(e) (i) Differentiate $x \log_e x$. $\boxed{1}$

(ii) Hence integrate $\log_e x$. $\boxed{2}$

(f) On a certain railway line, there are eleven railway stations at which a train can stop. The rail authority needs to print tickets for travel between every possible pair of stations on the line. How many different one-way tickets must be printed if the ticket specifies which direction the passenger is travelling? $\boxed{1}$

Marks

Question 2 (12 points)

(a) Find

(i) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2 x\, dx,$ $\boxed{2}$

(ii) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x\, dx.$ $\boxed{2}$

(b) (i) Prove by mathematical induction that $\boxed{4}$

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}.$$

(ii) What can you say about $\boxed{1}$

$$\lim_{n\to\infty} \left(\sum_{r=1}^{n} \frac{1}{r(r+1)} \right)?$$

(c) Using one iteration of Newton's Method and a first approximation of $x_0 = 0.7$, find, correct to three decimal places, a second approximation to $\boxed{3}$

$$y = \sin^3 x - 0.25.$$

Section B

(Use a separate writing booklet.)

Marks

Question 3 (12 points)

(a) By writing $\cot x$ as $\dfrac{\cos x}{\sin x}$, evaluate $\boxed{2}$

$$\int_{\pi/6}^{\pi/3} \cot x \, dx.$$

(b) Given that a team of five players is to be selected from a group of ten boys, find the number of teams that contain

(i) at least one of the two best players, $\boxed{2}$

(ii) no more than one of the three youngest players. $\boxed{3}$

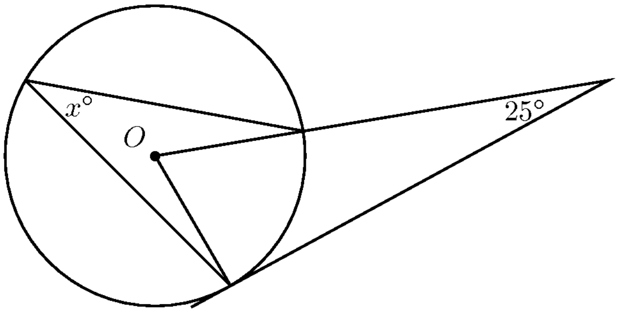
(c) (i) Show that $e^x = 3x + 2$ has a solution between 2 and 2.5. $\boxed{2}$

(ii) Hence use "halving the interval" to find, correct to one decimal place, a solution in the interval $[2,\ 2.5]$. $\boxed{3}$

Marks

Question 4 (12 points)

(a) 3

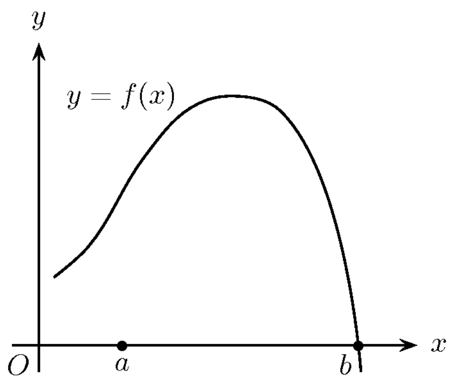


Find the value of x, giving reasons.

(b) Prove that $7^{n+1} + 3^n$ is divisible by 4 for all positive integers n. 4

(c) A team of three people is to be chosen from six men and five women by putting the eleven names in a hat and drawing out three simultaneously at random. Find the probability that the team will be of mixed sex. 3

(d) 2



Consider the above graph of $y = f(x)$. The value a shown on the axis is taken as the first approximation to the solution b of $f(x) = 0$.

Is the second approximation obtained by Newton's method a better approximation to b than a is? Give a reason for your answer.

Section C

(Use a separate writing booklet.)

Marks

Question 5 (12 points)

(a) Differentiate with respect to x

(i) $\log_e(\cos x)$, $\boxed{2}$

(ii) $(x+1)e^{-x}$. $\boxed{2}$

(b) (i) Differentiate $x + \log_e x$. $\boxed{2}$

(ii) Hence or otherwise, find a primitive of $\dfrac{x+1}{x^2 + x\log_e x}$. $\boxed{3}$

(c) AKB, CKD are two chords of a circle (meeting at an internal point K). $\boxed{3}$
Given the following lengths
$AB = 10\,\text{cm},\ CD = 6\,\text{cm},\ AK = 1\,\text{cm}$,
calculate the ratio $AC : BD$.

Marks

Question 6 (12 points)

(a) (i) How many three-figure numbers can be formed from the nine digits $1, 2, 3, \ldots 9$, there being no repetitions? $\boxed{1}$

(ii) From a pack of nine cards numbered $1, \ldots 9$, three cards are drawn at random and laid on a table from left to right.

(α) What is the probability that the number formed by the three digits drawn should exceed 500? $\boxed{1}$

(β) What is the probability that the digits should be drawn in ascending order, not necessarily consecutive? $\boxed{3}$

(b) (i) Prove that the graph of $y = \ln x$ is concave down for all $x > 0$. $\boxed{2}$

(ii) Sketch the graph of $y = \ln x$. $\boxed{1}$

(iii) Suppose $1 < a < b$ and consider the points $A(a, \ln a)$ and $B(b, \ln b)$ on the graph of $y = \ln x$. $\boxed{2}$

Find the coördinates of the point P that divides the line segment AB in the ratio $2 : 1$.

(iv) By using (ii) and (iii), deduce that $\frac{1}{3}\ln a + \frac{2}{3}\ln b < \ln\left(\frac{1}{3}a + \frac{2}{3}b\right)$. $\boxed{2}$

End of Paper

STANDARD INTEGRALS

$$
\begin{aligned}
\int x^n dx &= \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0 \\
\int \frac{1}{x} dx &= \ln x, \quad x > 0 \\
\int e^{ax} dx &= \frac{1}{a}e^{ax}, \quad a \neq 0 \\
\int \cos ax dx &= \frac{1}{a}\sin ax, \quad a \neq 0 \\
\int \sin ax dx &= -\frac{1}{a}\cos ax, \quad a \neq 0 \\
\int \sec^2 ax dx &= \frac{1}{a}\tan ax, \quad a \neq 0 \\
\int \sec ax \tan ax dx &= \frac{1}{a}\sec ax, \quad a \neq 0 \\
\int \frac{1}{a^2+x^2} dx &= \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0 \\
\int \frac{1}{\sqrt{a^2-x^2}} dx &= \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a \\
\int \frac{1}{\sqrt{x^2-a^2}} dx &= \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0 \\
\int \frac{1}{\sqrt{x^2+a^2}} dx &= \ln\left(x + \sqrt{x^2+a^2}\right)
\end{aligned}
$$

NOTE: $\ln x = \log_e x, \quad x > 0$

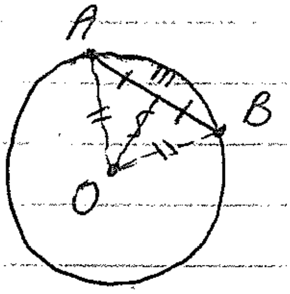
April 2010 Maths Ext 1 exam · YR12 Task 2.

12.

1 (a) (i) $4\times3\times2\times1 = 4! = 24$ ①

(ii) $\underline{3}\times\underline{2}\times\underline{1}\times\underline{3} = 18$ ①

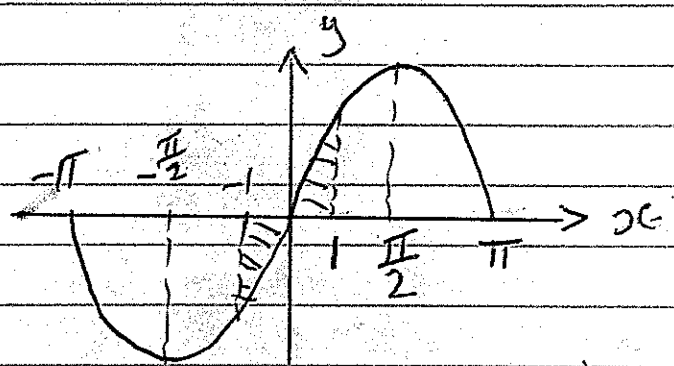
(b)



Join AO and BO
In $\triangle AOM$ and $\triangle BOM$
$AO = BO$ radii
$AM = MB$ data
OM common side
$\therefore \triangle AOM \equiv \triangle BOM$ (SSS)

Hence $A\hat{M}O + B\hat{M}O = 180^\circ$
But $A\hat{M}O = B\hat{M}O$ corresponding angles in congruent triangles
So $A\hat{M}O = B\hat{M}O = 90^\circ$
Hence line from centre of circle to the midpt of a chord is $\perp$ to the chord. ③

(c)



area under curve (shaded) cancels out area above curve (shaded)
$\sin x$ is an odd function ②

(d) (ii) $\lim_{x \to 0} \frac{\tan 4x}{7x}$

$\frac{4}{7} \lim_{x \to 0} \frac{\tan 4x}{4x} = \frac{4}{7} \times 1 = \frac{4}{7}$ 1

(e) (i) $\frac{d}{dx}(x \ln x) = x \times \frac{1}{x} + \ln x \times 1$

$= 1 + \ln x$ 1

so $\frac{d}{dx}(x \ln x) - 1 = \ln x$

(ii) $\int \left(\frac{d}{dx}(x \ln x) - 1 \right) dx = \int \ln x \, dx$

$\therefore \int \ln x \, dx = x \ln x - x + C$ 2

(f) $2 \times (10+9+8+7+6+5+4+3+2) = 110$ 1

or ${}^{11}P_2$

2 (a) (i) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2 x \, dx$ using $\tan^2 x = \sec^2 x - 1$

/12

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2 x - 1) \, dx$

$= \left[\tan x - x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$ 2

$= \left(\tan \frac{\pi}{3} - \frac{\pi}{3} \right) - \left(\tan \frac{\pi}{6} - \frac{\pi}{6} \right)$

$= \sqrt{3} - \frac{\pi}{3} - \frac{1}{\sqrt{3}} + \frac{\pi}{6} = \frac{2}{\sqrt{3}} - \frac{\pi}{6} = \left(\frac{2\sqrt{3}}{3} - \frac{\pi}{6} \right)$

(3)

2 (a) (ii) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 x \, dx$

$$= \left(\frac{1}{2}x - \frac{1}{4}\sin 2x \right)_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left(\frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3} \right) - \left(\frac{\pi}{12} - \frac{1}{4}\sin\frac{\pi}{3} \right)$$

$$= \frac{\pi}{6} - \frac{\pi}{12} - \frac{1}{4}\times\frac{\sqrt{3}}{2} + \frac{1}{4}\times\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{12}$$

(2)

(b) (i) m. Induction $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

LHS means: $\underset{r=1}{\frac{1}{1\times 2}} + \underset{r=2}{\frac{1}{2\times 3}} + \underset{r=3}{\frac{1}{3\times 4}} + \cdots + \frac{1}{n(n+1)} = \underset{\text{sum}}{\frac{n}{n+1}}$

Step 1 let $r=1$, LHS $= \frac{1}{1\times 2} = \frac{1}{2}$.

RHS, $n=1$ term $\frac{1}{1+1} = \frac{1}{2}$.

True for $n=1$

step 2 let there be a value of $n=k$, $(k \le n)$ such that $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$ is true

and we must prove that when $n=k+1$

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

LHS. $\frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)}$

LHS $\frac{(k+1)}{(k+2)}$

$= RHS$.

$\therefore n = k+1$ is true

4

step 3 By the principle of math. induction if it is true for $n=k$, it is true for $n=k+1$ etc

(b) (ii) as $n \to \infty$, $\lim_{n\to\infty}\left(\sum_{r=1}^{n} \frac{1}{r(r+1)}\right) = 1$

1

2 (c) $y = \sin^3 x - 0.25$

$y' = 3\sin^2 x \cos x$

Using $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$y = \sin^3 0.7 - 0.25 = 0.01736$

$y' = 3\sin^2 0.7 \times \cos 0.7 = 0.95227$

$x_1 = 0.7 - \frac{0.01736}{0.95227}$

$\doteq$ (3DP).

0.682

3

Extension 1 Solutions

SECTION B

Q3 (a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cot x \, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x} dx$

$= \left[\ln \sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \ln \sin \frac{\pi}{3} - \ln \sin \frac{\pi}{6}$

$= \ln \frac{\sqrt{3}}{2} - \ln \frac{1}{2}$

$= \ln \left(\frac{\sqrt{3}}{2} \div \frac{1}{2}\right)$ [using log laws]

$= \ln \sqrt{3}$

$= 0.549$ correct to 3 decimal places

(b)(i) Having at least one of the two best players means having one of the two or both of them

$\therefore$ No. teams $= {}^2C_1 \times {}^8C_4 + {}^2C_2 \times {}^8C_3$

$= 140 + 56$

$= 196.$

(ii) Having no more than one of the three youngest players means having none of the youngest players or having just one of them

$\therefore$ No. teams $= {}^7C_5 + {}^3C_1 \times {}^7C_4$

$= 21 + 105$

$= 126.$

Q13 (c) (i) Let $f(x) = e^x - 3x - 2$ and consider $f(x) = 0$.

Now $f(2) = e^2 - 6 - 2 = 7.4 - 6.2 = -0.6$

and $f(2.5) = e^{2.5} - 7.5 - 2 = 12.18 - 9.5 = 2.68$

Now $f(x)$ is continuous for $2 \le x \le 2.5$ and changes sign in that interval.

$\therefore f(x)$ has a zero between 2 and 2.5.

(ii) Consider $f\left(\frac{2+2.5}{2}\right) = f(2.25) = 9.488 - 8.75 = 0.734$

$\therefore$ There is a root between $x = 2$ and $x = 2.25$

Consider $f\left(\frac{2+2.25}{2}\right) = f(2.125) = e^{2.125} - 8.375 = -0.002$

$\therefore$ There is a root between $x = 2.125$ and $x = 2.25$

Consider $f\left(\frac{2.125+2.25}{2}\right) = f(2.188) \doteqdot 0.349$

$\therefore$ There is a root between $x = 2.88$ and $x = 2.125$

Consider $f\left(\frac{2.188+2.125}{2}\right) = f(2.157) \doteqdot 0.173$

$\therefore$ There is a root between $x = 2.157$ and $x = 2.125$

Consider $f\left(\frac{2.157+2.125}{2}\right) = f(2.141) \doteqdot 0.081$

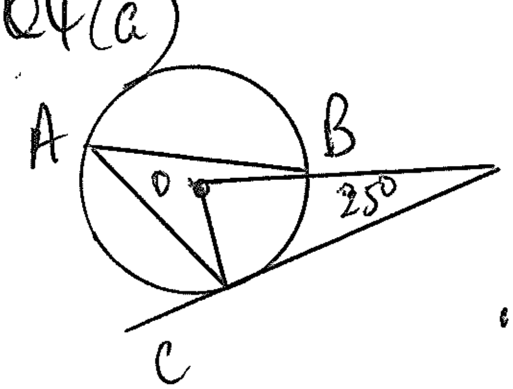
$\therefore$ There is a root between $x = 2.125$ and $x = 2.141$

And 2.125 and 2.141 are both 2.1 correct to one decimal place.

$\therefore$ No further application of halving the interval is required to find the required root

i.e. The root is 2.1.

Q4 (a)



From the figure as labelled.

$\angle OCD = 90°$ [tangent ⊥ to radius drawn to point of contact]

$\therefore \angle COD = 180 - (90 + 25)°$ [∠ sum of Δ is 180°]

$= 65°$.

$\therefore \angle CAB = \frac{65°}{2}$ [angle at centre 2 × angle at circumference on same arc]

$= 32\frac{1}{2}°$

(b) Consider $S(n) = 7^{n+1} + 3^n$

Step 1: Let $n = 1$, then $S(1) = 7^{1+1} + 3^1 = 49 + 3 = 52$

And $52 = 4 \times 13$

$\therefore$ The statement is true for $n = 1$. —— (A)

Step 2: Assume that the statement is true for $n = k$

i.e Assume that $7^{k+1} + 3^k = 4A$ (A an integer)

Now consider the statement for $n = k+1$.

Then $S(k+1) = 7^{k+2} + 3^{k+1}$

$= 7.7^{k+1} + 3^{k+1}$

$= 4.7^{k+1} + 3.7^{k+1} + 3.3^k$

$= 4.7^{k+1} + 3(7^{k+1} + 3.3^k)$

$= 4 + 3.4A$ using the assumption

$= 4(1 + 3A)$ and $1+3A$ is integral.

$\therefore$ The statement is true for $n = k+1$ if it is true for $n = k$ as assumed.

Step 3: But the statement is true for $n = 1$ (from A)

$\therefore$ True for $n = 1+1 = 2$. $\Rightarrow$ true for $2+1 = 3$

and so on for all integral n.

Q4 (c) No. of teams possible without restriction $= {}^{11}C_3$

$= 165$

No. of teams with mixed sex = No. teams 2M & 1W + No. Teams 1M & 2W

$= {}^6C_2 \times {}^5C_1 + {}^6C_1 \times {}^5C_2$

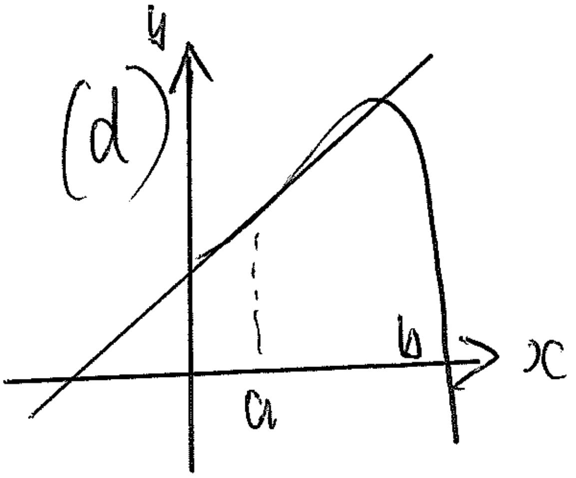
$= 75 + 60$

$= 135.$

$\therefore P(E) = \frac{135}{165}$

$= \frac{9}{11}.$

(d)



Newton's method would not give a better approximation because there is a turning point between a and b.

Thus the tangent to the curve at a will cut the x-axis further away from b than a is.

Section C

5) a) i) $y = \log_e(\cos x)$

$$y' = \frac{-\sin x}{\cos x}$$

$$y' = -\tan x$$

(ii) $y = (x+1)e^{-x}$ $\quad u = x+1 \quad v = e^{-x}$
$\quad u' = 1 \quad v' = -e^{-x}$

$$y' = -(x+1)e^{-x} + e^{-x}$$

$$y' = -xe^{-x} - e^{-x} + e^{-x}$$

$$y' = -xe^{-x}$$

(b) i) $$\frac{d(x+\log_e x)}{dx} = 1 + \frac{1}{x}$$

$$= \frac{x+1}{x}$$

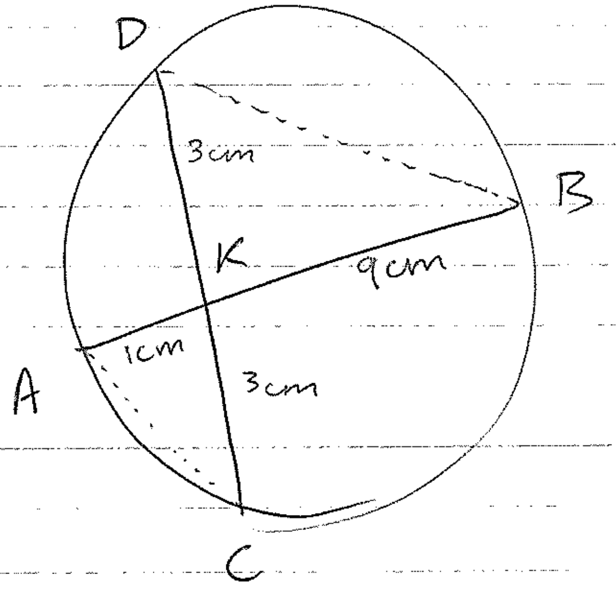
(ii) $$\int \frac{x+1}{x^2+x\log_e x}\,dx = \int \frac{x+1}{x(x+\log_e x)}\,dx$$

$$= \int \frac{\left(\frac{x+1}{x}\right)}{x+\log_e x}\,dx$$

in form $\int \frac{f'(x)}{f(x)}\,dx$

$$= \log_e(x+\log_e x) + C$$

(c)



$CD = 6cm.$

$AK.KB = CK.KD$
(product of intercepts, intersecting chords)

let $CK = x$

$1 \times 9 = x(6-x)$

$6x - x^2 = 9$

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

$x = 3.$

In $\triangle$'s AKC & DKB

$\angle AKC = \angle DKB$ (vertically opposite angles)

$\angle CAK = \angle BDK$ (angles in same segment)

$\therefore \triangle AKC \,|||\, \triangle DKB$ (equiangular)

$\frac{AC}{BD} = \frac{AK}{DK}$ (corresponding sides of similar triangles in same ratio)

$\frac{AC}{BD} = \frac{1}{3}$

$\therefore AC : BD = 1 : 3$

6) a) i) $9 \times 8 \times 7 = 504$ or ${}^9P_3$

(ii) α) $5 \times 8 \times 7 = 280$

Probability $= \frac{280}{504}$

$= \frac{5}{9}$

(β) Given a 3 digit number (of different digits not including zero), only one arrangement will be in ascending order.

ie $\frac{1}{3!} = \frac{1}{6}$ or $\frac{{}^9C_3}{{}^9P_3}$

(b) i) $y = \ln x$, $x > 0$

$y' = \frac{1}{x} = x^{-1}$

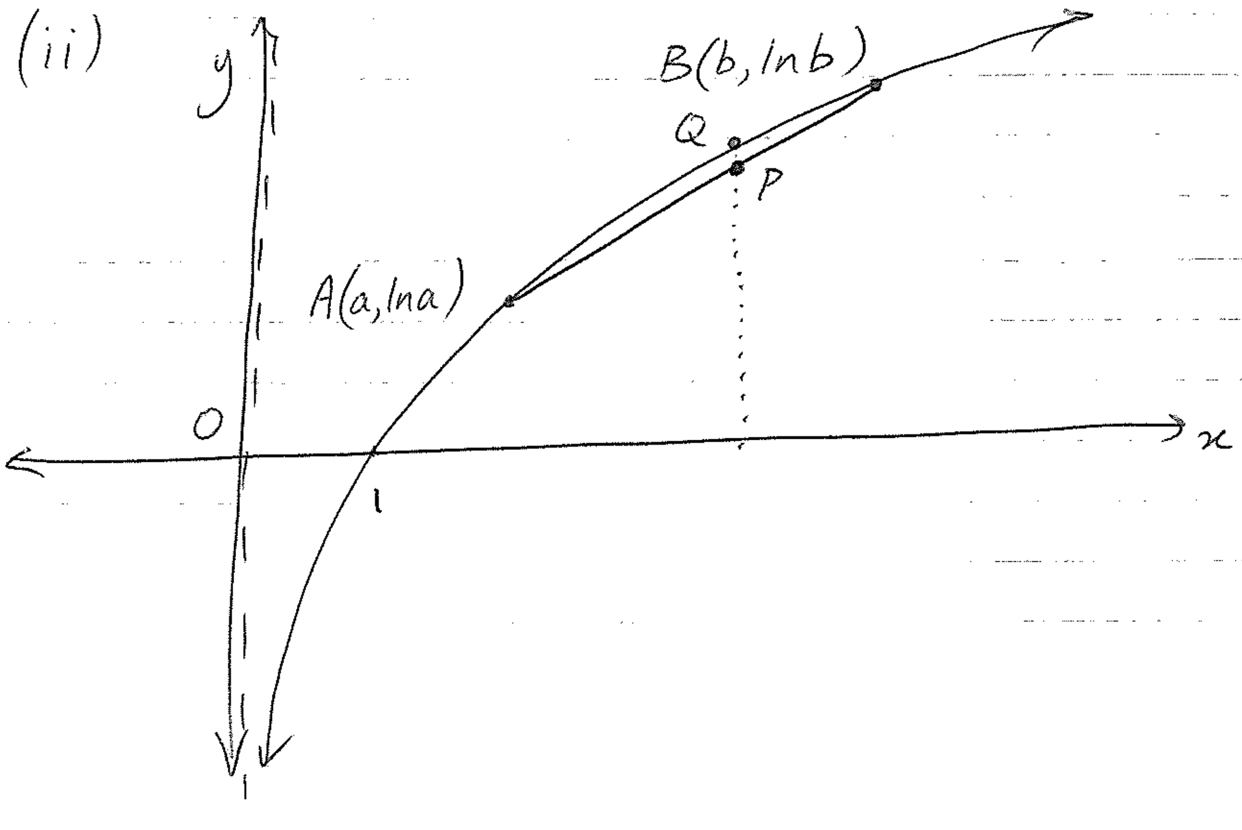
$y'' = -x^{-2}$

$= -\frac{1}{x^2}$

$y'' < 0$ for all $x > 0$

$\therefore y = \ln x$ is concave down for all $x > 0$.

(ii)



(iii) $A(a, \ln a)$ $B(b, \ln b)$

$2 : 1$

$$P = \left(\frac{a + 2b}{2+1}, \frac{\ln a + 2\ln b}{2+1} \right)$$

$$P = \left(\frac{1}{3}a + \frac{2}{3}b, \frac{1}{3}\ln a + \frac{2}{3}\ln b \right)$$

(iv) Consider the point Q which lies on $y = \ln x$ with the same x-coordinate as P.

Clearly from the graph in (ii)

$$y_P < y_Q.$$

$$\frac{1}{3}\ln a + \frac{2}{3}\ln b < \ln\left(\frac{1}{3}a + \frac{2}{3}b \right)$$