SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

April 2014

Assessment Task 2
Year 12

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question if full marks are to be awarded.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 80

- Attempt sections A – D.
- Start each **NEW** section in a separate answer booklet.
- Hand in your answers in 4 separate bundles:
  Section A
  Section B
  Section C
  Section D

Examiner: *J. Chen*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

START A NEW ANSWER BOOKLET

SECTION A [20 marks] **Marks**

1. If $f(x) = x + 5$, find the inverse function $f^{-1}(x)$. **[1]**

2. Find the Cartesian equation whose parametric equations are **[1]**

$$x = 8t, \qquad y = 4t^2$$

3. If α, β, γ are the roots of the equation $x^3 + x^2 - 2x - 1 = 0$. Find the value of

   (i) $\alpha + \beta + \gamma$ **[1]**

   (ii) $\alpha\beta\gamma$ **[1]**

   (iii) $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$ **[1]**

4. Let $y = e^x$. Find $\frac{dy}{dx}$ when $x = 1$. **[2]**

5. Find **[2]**

$$\frac{d}{dx}\cos^{-1} 2x$$

6. There are 40 Students in a class. 38 of them are boys. If 5 students are randomly selected, find the number of ways that at least 1 girl is selected. **[2]**

7. Let $P(x)$ be a polynomial. When $P(x)$ is divided by $x - 2$, the quotient is $6x^2 + 5x + 9$. It is given that $P(2) = 20$.

   (i) Find $P\left(\frac{2}{3}\right)$. **[2]**

   (ii) Solve $P(x) = 0$ **[3]**

8. Peter invites 8 friends to join his birthday party. There are 5 boys and 4 girls at the party including Peter. In a game, boys and girls sit in a row. What is the probability that the boys and girls sit alternatively? **[2]**

9. Find **[2]**

$$\int \frac{2}{\sqrt{16 - x^2}}.dx$$

End of Section A

START A NEW ANSWER BOOKLET

SECTION B [20 marks] **Marks**

1. Let $P(x) = x^3 + x^2 + mx + n$. If $P(1) = 0$ and the remainder of $P(x)$ divided by $x - 2$ is 11, then factorise $P(x)$. **[3]**

2. Consider the function $f(x) = \sin^{-1}(x - 2)$, evaluate $f\left(\frac{3}{2}\right)$. **[2]**

3. There are 24 boys and 4 girls in a class. From the class, 5 students are randomly selected to form the class committee.

   (i) Find the probability that the class committee consists of boys only. **[1]**

   (ii) Find the probability that the class committee consists of at least 2 boys and 1 girl. **[3]**

4. Show that if $t = \tan x$, then **[2]**

$$\frac{dx}{dt} = \frac{1}{1 + t^2}$$

5. Find

   (i)
$$\int \frac{1}{\sqrt{25 - 9x^2}}.dx$$
[2]

   (ii)
$$\int \frac{x + 1}{x^2 + 1} dx$$
[2]

6. Find **[2]**

$$\int (e^x + e^{-x})^2 . dx$$

7. Find the volume of the solid generated when the region bounded by the curve $y = e^x$ and the x-axis in the interval $0 \leq x \leq 2$ is rotated about the x-axis. **[3]**

End of Section B

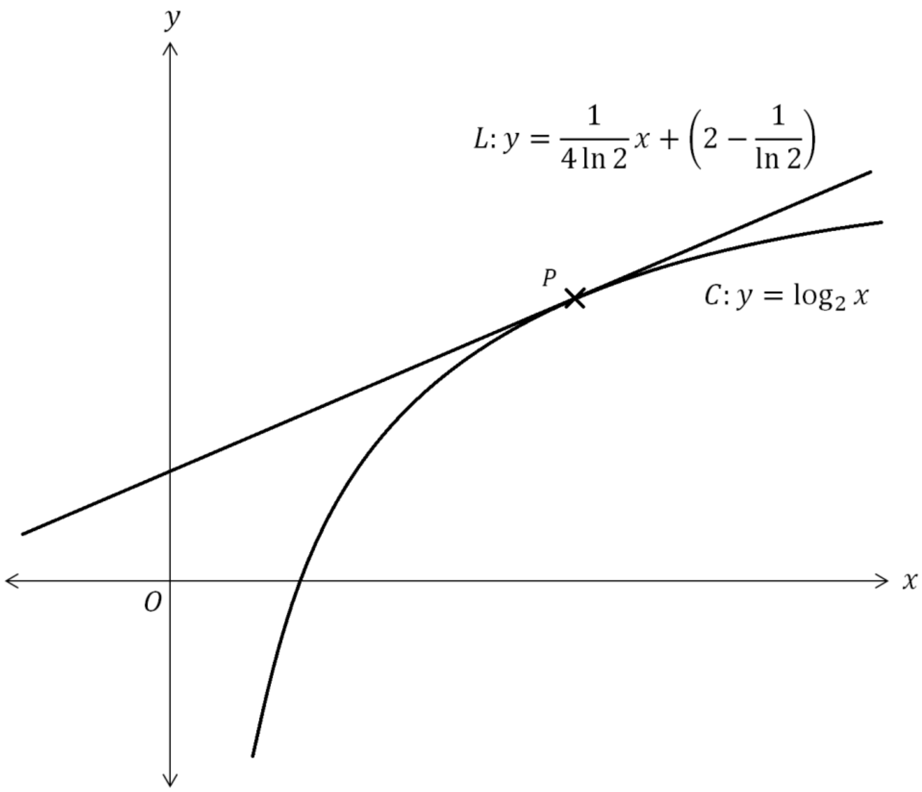
START A NEW ANSWER BOOKLET

SECTION C [20 marks] **Marks**

1. Find **[2]**

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{3-4x^2}}$$

2. In the diagram below, the line $L: y = \frac{1}{4\ln 2}x + \left(2 - \frac{1}{\ln 2}\right)$ touches the curve $C: y = \log_2 x$ at the point P. Find the coordinates of P. **[3]**



3. Prove, by mathematical induction, that $9^n - 1$ is divisible by 8 for all positive integers n. **[3]**

4. The slope at any point (x, y) of a curve is given by $\frac{dy}{dx} = 2e^{2x} - 3e^x$. The y-intercept of the curve is 0. Find the area bounded by the curve and the x-axis. **[4]**

Section C continues on next page

5.

(i) Find $\frac{d}{dx}(x^2 \ln x)$ **[2]**

(ii) Hence, find **[3]**

$$\int x \ln x \, . \, dx$$

6. Sketch $y = 3\cos^{-1} 2x$. State the domain and range. **[3]**

End of Section C

START A NEW ANSWER BOOKLET

SECTION D [20 marks] **Marks**

1. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$. The tangents at P and Q meet at the point T.

   (i) Show that the equation of the tangent at P is $y = px - ap^2$ and similarly, write down the equation of the tangent at Q. **[2]**

   (ii) Find the coordinates of T. **[2]**

   (iii) Let M be the midpoint of PQ, find the coordinates of M. **[1]**

   (iv) The point R is the midpoint of MT, show that R has coordinates **[1]**

$$\left[a(p+q), \frac{a(p+q)^2}{4}\right]$$

   (v) Find the Cartesian equation of the locus of R and describe the locus in geometrical terms. **[2]**

2. Consider the following letters **[3]**

$$A, A, B, B, C, C$$

   how many different arrangements are possible if no two identical letters are next to one another?

3.

   (i) Show that **[2]**

$$a^n + b^n = (a+b)(a^{n-1} + b^{n-1}) - ab(a^{n-2} + b^{n-2}),$$

   where a and b are real numbers.

   (ii) Let $x + \frac{1}{x} = 2\cos\theta$, where $\theta \neq \pi$ and $x \neq 0$. Suppose $x^k + \frac{1}{x^k} = 2\cos k\theta$ and $x^{k+1} + \frac{1}{x^{k+1}} = 2\cos(k+1)\theta$ for a positive integer k. **[3]**

   Show that $x^{k+2} + \frac{1}{x^{k+2}} = 2\cos(k+2)\theta$.

   (iii) It is given that $x^n + \frac{1}{x^n} = 2\cos n\theta$. Prove, by mathematical induction, that **[4]**

$$(x + x^3 + x^5 + \cdots + x^{2n-1}) + \left(\frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^5} + \cdots + \frac{1}{x^{2n-1}}\right) = \frac{\sin 2n\theta}{\sin\theta}$$

   for all positive integers n, where $\theta \neq k\pi$.

End of Section D
End of Exam

EXT 1 ASS 2 2014

Section A

1. $f(x) = x+5$

ie $y = x+5$

For $f^{-1}(x)$:

$x = y+5$

ie $y = x-5$

As $f^{-1}(x) = x-5$ [1]

2. $x = 8t$; $y = 4t^2$

$t = \frac{x}{8}$; $y = 4\left(\frac{x}{8}\right)^2$

$= \frac{4x^2}{64}$

$\therefore y = \frac{x^2}{16}$ [1]

3. $x^3+x^2-2x-1=0$

(i) $\alpha+\beta+\gamma = -1$ [1]

(ii) $\alpha\beta\gamma = 1$ [1]

(iii) $\alpha^2\beta\gamma + \alpha\beta^2\cancel{\gamma} + \alpha\beta\gamma^2$

$= \alpha\beta\gamma(\alpha+\beta+\gamma)$

$= +1 \times -1$

$= -1$ [1]

4. $y = e^x$

$\frac{dy}{dx} = e^x$

When $x=1$

$\frac{dy}{dx} = e^1$

$= e$ [2]

5. $\frac{d}{dx}\cos^{-1}2x$

$= \frac{-1}{\sqrt{1-(2x)^2}} \times 2$

$= \frac{-2}{\sqrt{1-4x^2}}$ [2]

6. No. of ways of choosing 5 including at least one girl is the total no. of ways of choosing 5, minus the no. of ways of choosing 5 boys.

${}^{40}C_5 - {}^{38}C_5$

$= 156066$

[2]

7. Division transformation:

$P(x) = (x-2)(6x^2+5x+9) + 20$ [2]

(i) $P\left(\frac{2}{3}\right) = 0$

(ii) $P(x) = 6x^3 - 7x^2 - x + 2$

From (i) $\left(x - \frac{2}{3}\right)$ is a factor

$P(1) = 0$

$\therefore (x-1)$ is a factor

Now $\alpha\beta\gamma = \frac{-2}{6}$

$= -\frac{1}{3}$

$\therefore \frac{2}{3} \times 1 \times \gamma = -\frac{1}{3}$

$\gamma = -\frac{1}{3} \times \frac{3}{2}$

$= -\frac{1}{2}$

$\therefore P(x) = 0$ for $x = -\frac{1}{2}, \frac{2}{3}, 1$

[3]

8.

B		B		B		B		B

Probability $= \frac{5! \times 4!}{9!}$

$= \frac{1}{126}$

[2]

9. $\int \frac{2}{\sqrt{16-x^2}}\,dx$

$= 2\int \frac{dx}{\sqrt{4^2-x^2}}$

$= 2\sin^{-1}\frac{x}{4} + C$

[2]

(X1) SECTION B

(1) $P(1) = 0 \Rightarrow 1 + 1 + m + n = 0$

ie $\boxed{m + n = -2}$ — (1)

also $P(2) = 11 \Rightarrow 8 + 4 + 2m + n = 11$

$\therefore \boxed{2m + n = -1}$ (2)

From (1) + (2)

$$\boxed{\begin{array}{l} m = 1 \\ n = -3 \end{array}}$$

$\therefore P(x) = x^3 + x^2 + x - 3$

now $(x-1)$ is a factor.

$\therefore \boxed{P(x) = (x-1)(x^2 + 2x + 3)}$

$$\begin{array}{r} x^2 + 2x + 3 \\ x - 1 \,\big)\, \overline{x^3 + x^2 + x - 3} \\ \underline{x^3 - x^2} \\ 2x^2 + x \\ \underline{2x - 2x} \\ 3x - 3 \end{array}$$

(2.) $f(x) = \sin^{-1}(x-2)$

$\therefore f(\frac{3}{2}) = \sin^{-1}(\frac{3}{2} - 2)$

$= \sin^{-1}(-\frac{1}{2})$

$= \boxed{-\frac{\pi}{6}}$

NB $|\sin^{-1}\theta| \leq \frac{\pi}{2}$

(3) (i) $\dfrac{\binom{24}{5}}{\binom{28}{5}} = \boxed{\dfrac{253}{585}}$

(ii) $\dfrac{\binom{24}{4}\times\binom{4}{1} + \binom{24}{3}\times\binom{4}{2} + \binom{24}{2}\times\binom{4}{3}}{\binom{28}{5}}$

$= \boxed{\dfrac{2323}{4095}}$

(4) $t = \tan x$

$\therefore x = \tan^{-1} t$

$\boxed{\dfrac{dx}{dt} = \dfrac{1}{1+t^2}.}$

(5) (i) $\displaystyle\int \frac{dx}{\sqrt{25-9x^2}} = \int \frac{dx}{\sqrt{9\left(\frac{25}{9} - x^2\right)}}$

$\displaystyle = \frac{1}{3}\int \frac{dx}{\sqrt{(5/3)^2 - x^2}}$

$\boxed{= \frac{1}{3}\sin^{-1}\frac{3x}{5} + C}$

(ii) $\displaystyle\int \frac{x+1}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{dx}{x^2+1}$

$\boxed{= \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C}$

(6) $$\int (e^x + e^{-x})^2 dx = \int (e^{2x} + 2e^0 + e^{-2x}) dx$$

$$= \int (e^{2x} + 2 + e^{-2x}) dx$$

$$\boxed{= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c}$$

(7). $$V = \pi \int_0^2 (e^x)^2 dx.$$

$$= \pi \int_0^2 e^{2x} dx$$

$$= \pi \left[\frac{1}{2}e^{2x}\right]_0^2$$

$$= \frac{\pi}{2}(e^4 - e^0)$$

$$\boxed{= \frac{\pi}{2}(e^4 - 1) \quad u^3.}$$

Section C (20 marks)

1. $\displaystyle\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{3-4x^2}}$

$= \displaystyle\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{4(3/4 - x^2)}}$

$= \frac{1}{2}\displaystyle\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{3/4 - x^2}}$

$= \frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right]_0^{\sqrt{3}/2}$

$= \frac{1}{2}\left(\sin^{-1} 1 - \sin^{-1} 0\right)$

$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$

$= \pi/4$.

2

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2. $y = \left(\frac{1}{\ln 2}\right)\ln x$ is curve

$y' = \frac{1}{\ln 2}\cdot\frac{1}{x}$

At $P(x_P, y_P)$ $y' = \frac{1}{(\ln 2)x_P}$ is gradient of curve at P.

But gradient of line $= \frac{1}{4\ln 2}$

$\therefore \frac{1}{(\ln 2)x_P} = \frac{1}{4\ln 2}$

$\Rightarrow x_P = 4$ Sub into $C \Rightarrow$ $y_P = \frac{1}{\ln 2}\ln 4$

$y_P = 2$

$\therefore P = (4, 2)$

3

3. Prove $(9^n - 1)$ is divisible by 8, $n \in \mathbb{Z}^+$

Let $n=1$, then $9^1 - 1 = 8$ which is divis. by 8

$\therefore$ true for $n=1$

Assume true for $n=k$

ie. Let $9^k - 1 = 8m$ $\exists m \in \mathbb{Z}^+$

Let $n = k+1$ Must show $9^{k+1} - 1 = 8p$ $\exists p \in \mathbb{Z}$

Now $9^{k+1} - 1$

$= 9^k \cdot 9 - 1$

$= 9(9^k - 1) + 8$

$= 9(8m) + 8$ from assumption

$= 8(9m + 1)$

$= 8p$. $\exists p \in \mathbb{Z}^+$

$\therefore$ true for $n = k+1$

Therefore, by the Principle of Mathematical Induction, statement is true for all n, $n \in \mathbb{Z}^+$.

3

$\frac{dy}{dx} = 2e^{2x} - 3e^{x}$ and $y\text{-int} = (0,0)$

Then $y = e^{2x} - 3e^{x} + c$

$(0,0) \Rightarrow 0 = 1 - 3 + c$

$\Rightarrow c = 2$

$\therefore y = e^{2x} - 3e^{x} + 2$

For x-intercepts, let $y = 0$

$\Rightarrow e^{2x} - 3e^{x} + 2 = 0$

$e^{x} \quad -2$

$e^{x} \quad -1$

$(e^{x} - 2)(e^{x} - 1) = 0$

$e^{x} = 1$ or $e^{x} = 2$

$x = 0$ or $x = \ln 2$

$\therefore \text{Area} = \left| \int_0^{\ln 2} (e^{2x} - 3e^{x} + 2)\,dx \right|$

$= \left| \left[\frac{e^{2x}}{2} - 3e^{x} + 2x \right]_0^{\ln 2} \right|$

$= \left| (2 - 6 + 2\ln 2) - (\tfrac{1}{2} - 3 + 0) \right|$

$= \left| -\tfrac{3}{2} + \ln 4 \right|$

$= \tfrac{3}{2} - \ln 4$ square units

4

5. (i) $y = x^2 \ln x$

$y' = x^2 \frac{1}{x} + (\ln x) 2x$

$= x + 2x \ln x.$

2

(ii) Now $\int (x + 2x \ln x)\,dx = x^2 \ln x + c$

$\Rightarrow \frac{x^2}{2} + 2\int x \ln x\,dx = x^2 \ln x + c$

$\int x \ln x\,dx = \frac{1}{2}\left(x^2 \ln x - \frac{x^2}{2}\right) + c$

$\therefore \int (x \ln x)\,dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$

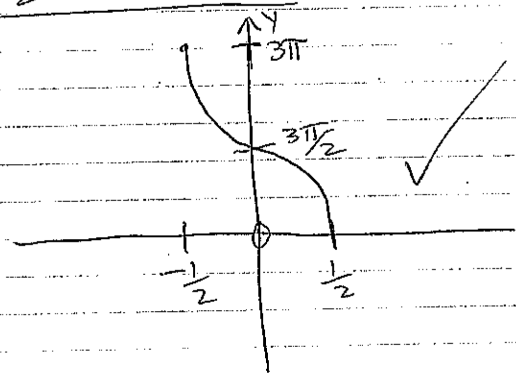
2

6. $y = 3\cos^{-1} 2x$

$\Rightarrow \frac{y}{3} = \cos^{-1} 2x$

D: $-1 \le 2x \le 1$ and Range: $0 \le \frac{y}{3} \le \pi$

D: $-\frac{1}{2} \le x \le \frac{1}{2}$ $\quad 0 \le y \le 3\pi$



3

SECTION D

1 (i) $x^2 = 4ay$

$\therefore 2x = 4ay'$

$\therefore y' = \dfrac{x}{2a}$

At P, $y' = \dfrac{2ap}{2a}$

$= p$

$\therefore$ Eqn of tangent is

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

Eqn of tangent at Q is

$y = qx - aq^2$ 2

(ii) $px - ap^2 = qx - aq^2$

$\therefore (p-q)x = ap^2 - aq^2$

$\therefore (p-q)x = a(p-q)(p+q)$

$\therefore x = a(p+q)$

$\therefore y = p(a(p+q)) - ap^2$

$= apq$

$\therefore$ T is $(a(p+q), apq)$ 2

(iii) M is $\left(\dfrac{2ap+2aq}{2}, \dfrac{ap^2+aq^2}{2}\right)$

$= \left(a(p+q), \dfrac{a(p^2+q^2)}{2}\right)$ 1

(iv) R is $\left(\dfrac{a(p+q)+a(p+q)}{2}, \dfrac{apq + \dfrac{a(p^2+q^2)}{2}}{2}\right)$

$= \left(a(p+q), \dfrac{a(2pq+p^2+q^2)}{4}\right)$

$= \left(a(p+q), \dfrac{a(p+q)^2}{4}\right)$ as given 1

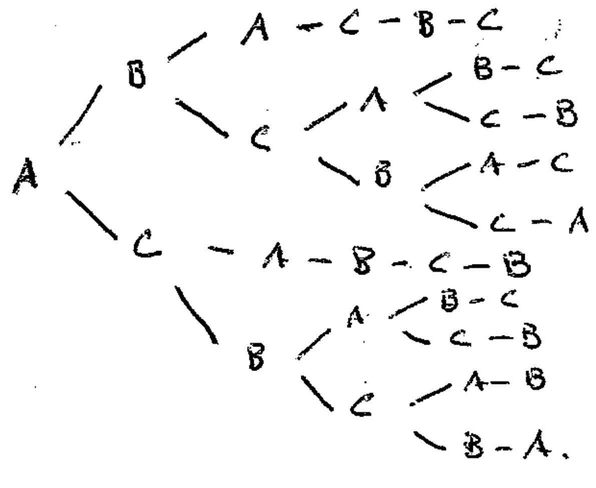
(v) Let $x = a(p+q)$

$\therefore y = \dfrac{x^2}{a} \cdot \dfrac{1}{4}$

$x^2 = 4ay$ 2

$\therefore$ R lies on the parabola $x^2 = 4ay$

2



10 arrangements.

$\therefore$ Total arrangements

$= 10 \times 3$

$= 30$ 3

3 (i) $(a+b)(a^{n-1}+b^{n-1}) - ab(a^{n-2}+b^{n-2})$

$= a^n + ab^{n-1} + ba^{n-1} + b^n - a^{n-1}b - ab^{n-1}$

$= a^n + b^n$ 2

(ii) $x^{k+2} + \frac{1}{x^{k+2}} = \left(x+\frac{1}{x}\right)\left(x^{k+1}+\frac{1}{x^{k+1}}\right) - 1.\left(x^k+\frac{1}{x^k}\right)$

$= 2\cos\theta . 2\cos(k+1)\theta - 2\cos k\theta$

$= 2\cos\theta(2\cos k\theta\cos\theta - 2\sin k\theta\sin\theta) - 2\cos k\theta$

$= 4\cos k\theta\cos^2\theta - 4\sin k\theta\sin\theta\cos\theta - 2\cos k\theta$

$= 2\cos k\theta(2\cos^2\theta - 1) - 2\sin k\theta\sin 2\theta$

$= 2(\cos k\theta\cos 2\theta - \sin k\theta\sin 2\theta)$

$= 2\cos(k+2)\theta$ 3

(iii) $S(n) \equiv (x+x^3+\dots+x^{2n-1}) + \left(\frac{1}{x}+\frac{1}{x^3}+\dots+\frac{1}{x^{2n-1}}\right)$

$= \frac{\sin 2n\theta}{\sin\theta}$

Step 1: Show $S(1)$ is true

i.e. $x + \frac{1}{x} = \frac{\sin 2\theta}{\sin\theta}$

LHS $= 2\cos\theta$

RHS $= \frac{2\sin\theta\cos\theta}{\sin\theta}$

$= 2\cos\theta$

$\therefore S(1)$ is true

Step 2: Assume $S(k)$ is true

i.e. $(x+x^3+\dots+x^{2k-1}) + \left(\frac{1}{x}+\frac{1}{x^3}+\dots+\frac{1}{x^{2k-1}}\right)$

$= \frac{\sin 2k\theta}{\sin\theta}$

Show $S(k+1)$ is true

i.e. $(x+x^3+\dots+x^{2k+1}) + \left(\frac{1}{x}+\frac{1}{x^3}+\dots+\frac{1}{x^{2k+1}}\right)$

$= \frac{\sin(2k+1)\theta}{\sin\theta}$

LHS $= \frac{\sin 2k\theta}{\sin\theta} + x^{2k+1} + \frac{1}{x^{2k+1}}$

$= \frac{\sin 2k\theta}{\sin\theta} + 2\cos(2k+1)\theta$

$= \frac{\sin 2k\theta + 2\cos(2k+1)\theta\sin\theta}{\sin\theta}$

$= \frac{\sin 2k\theta + 2\sin\theta(\cos 2k\theta\cos\theta - \sin 2k\theta\sin\theta)}{\sin\theta}$

$= \frac{\sin 2k\theta + \cos 2k\theta . 2\sin\theta\cos\theta - \sin 2k\theta . 2\sin^2\theta}{\sin\theta}$

$= \frac{\cos 2k\theta . \sin 2\theta + \sin 2k\theta(1-2\sin^2\theta)}{\sin\theta}$

$= \frac{\cos 2k\theta\sin 2\theta + \sin 2k\theta\cos 2\theta}{\sin\theta}$

$= \frac{\sin(2k+2)\theta}{\sin\theta}$

$=$ RHS

$\therefore$ if $S(k)$ is true, $S(k+1)$ is true

Step 3: $S(1)$ is true and $S(k+1)$ is true if $S(k)$ is true.

$\therefore$ By the process of Mathematical Induction $S(n)$ is true for all integers $n \geq 1$. 4