Name: File

Teacher/Class: ____________

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12

HSC ASSESSMENT TASK 2
MARCH 2006

EXTENSION 1 MATHEMATICS

Time Allowed: **70 minutes**

Instructions:

- Write your name and class at the top of each page.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start each question on a new page.
- Standard integrals can be found on the last page.

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total
/10	/10	/10	/10	/10	/10	/60

Question 1 **(10 marks)** **Marks**

a) Find the exact value of i. $\tan(\frac{2\pi}{3})$ 1

ii. $\sin(-\frac{\pi}{3})$ 1

b) Find

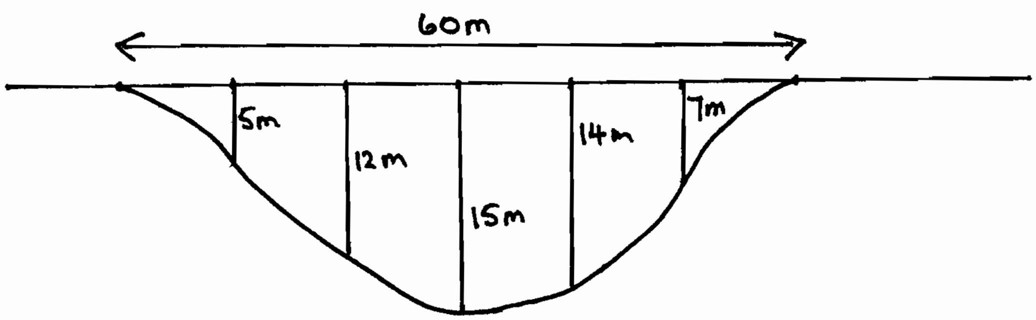
$$\lim_{x \to 0} \frac{\sin 2x}{x}$$ 1

c) Given that $\int_1^5 f(x)dx = 4$ find the value of k 3

for which $\int_1^5 [f(x) + kx\ ]\, dx = 28$.

d) A river 60m wide is surveyed for its depth every 10m across its width.

The depth at each point surveyed is shown on the diagram.

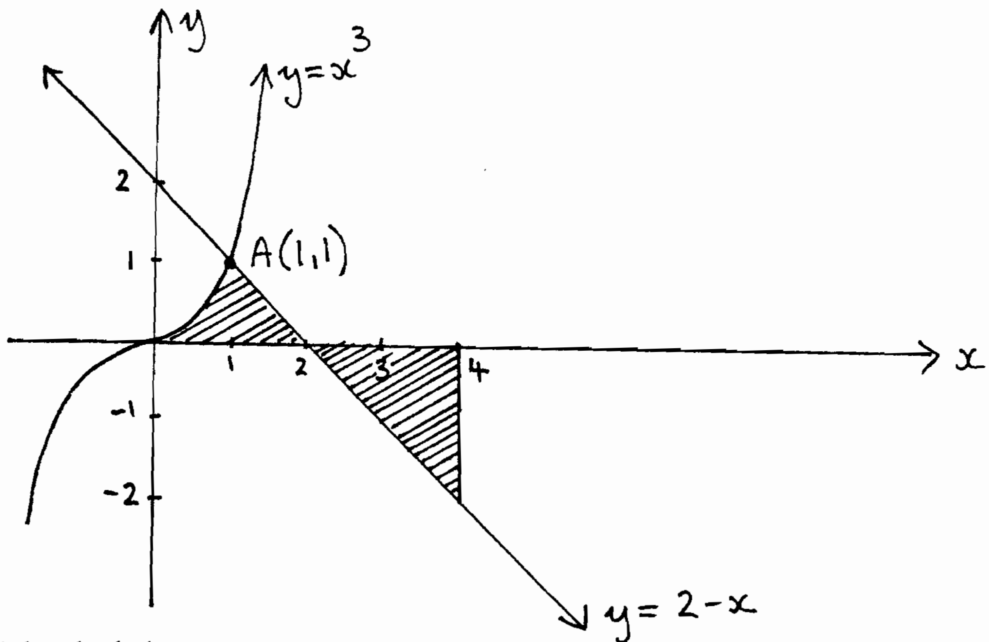


i. Find the cross-sectional area of the river using Simpsons Rule 3

ii. Hence find the volume of water passing this point per second if the water flows at 5m/s. 1

Question 2 **(10 marks) Start a new page** **Marks**

a) The point of intersection of $y = x^3$ and $y = 2 - x$ is the point A (1, 1) 3



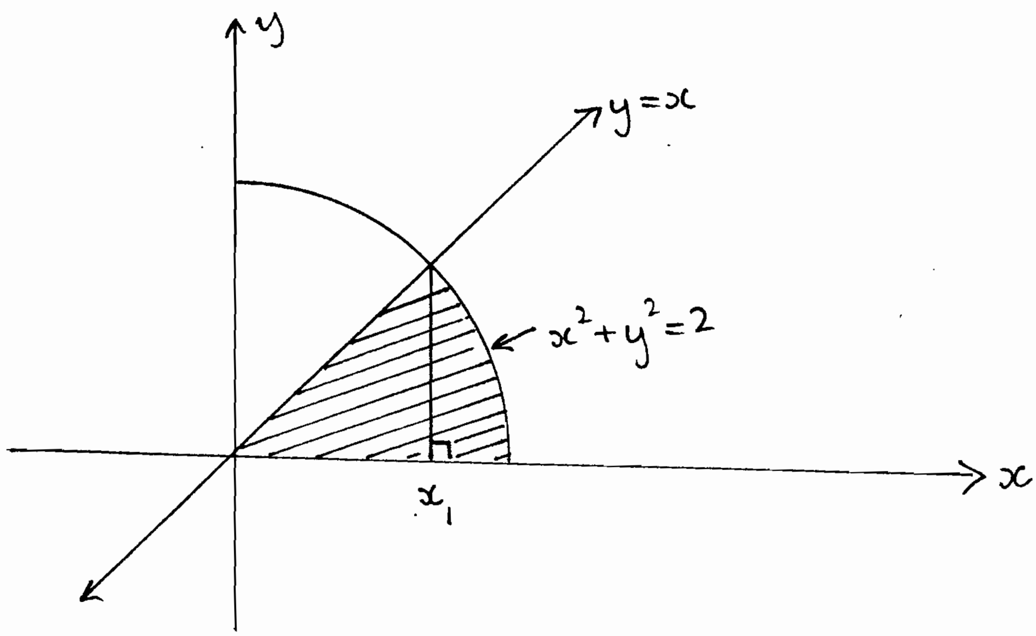
Find the shaded area.

b) If $y = \sin 2x°$

i. Express $2x°$ in radian measure 1

ii. Find $\int \sin 2x° \, dx$ 2

c)



i. Find x_1 1

ii. Calculate the volume generated when the shaded region (shown above) between the line $y = x$, the circle $x^2 + y^2 = 2$ and the x axis is rotated around the x axis. 3

Question 3 **(10 marks)** **Start a new page**

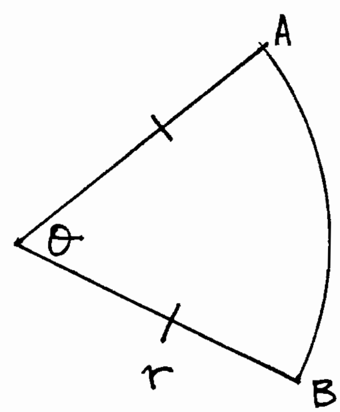
a) If $y = a\cos nx + b\sin nx$

show that $\dfrac{d^2y}{dx^2} + n^2y = 0$ 3

b) i. Differentiate $x\sqrt{x+3}$ and simplify your answer as far as possible. 2

ii. Hence find $\int \dfrac{x+2}{\sqrt{x+3}}dx$ 1

c) The sector below has area of $25cm^2$. It is contained in a circle of radius r cm and the arc AB subtends an angle at the centre of the circle of θ radians.



i. Show the perimeter of the sector is given by $P = 2r + \dfrac{50}{r}$ 1

ii. Find r for which the perimeter is a minimum. 3

Question 4 (10 marks) Start a new page

a) i. Sketch $y = 3\cos 2x$ for $0 \le x \le \pi$ 2

ii. Find the area enclosed by $y = 3\cos 2x$, the x axis, $x = 0$

and $x = \dfrac{\pi}{2}$ 3

b) i. Express $\sin x + \sqrt{3}\cos x$ in the form $A\sin(x + \theta)$ for $0 < \theta < \dfrac{\pi}{2}$ 2

ii. Hence solve $\sin x + \sqrt{3}\cos x = \sqrt{2}$ for $0 \le x \le 2\pi$. 3

Question 5 (10 marks) Start a new page

a) Find $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ using the substitution $u = \sqrt{x}$ 3

b) Find $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$ using the substitution $u = 1 - x$. 4

c) Find $\displaystyle\int \cos^2 3x \quad dx$ 3

Question 6 (10 marks) Start a new page

a) Evaluate $\displaystyle\int_0^1 \frac{x}{(x^2+2)^2}\,dx$ using the substitution $u = x^2 + 2$ 3

b) i. Prove $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$ 1

ii. Hence or otherwise evaluate $\displaystyle\int_0^{\pi/6} \sin 4x.\cos 2x \;\; dx$ 3

c) i. Sketch $y = 2^x$ 1

ii. If n is a <u>positive integer</u>, by considering the graph of $y = 2^x$ 2

show that $2^n < \displaystyle\int_n^{n+1} 2^x\,dx < 2.2^n$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

a) i) $\tan\left(\frac{2\pi}{3}\right) = \tan\left(\pi - \frac{\pi}{3}\right)$

$= -\tan \pi/3$

$= -\sqrt{3}$ (1)

ii) $\sin\left(-\frac{\pi}{3}\right) = \sin\left(2\pi - \frac{\pi}{3}\right)$

$= -\sin\frac{\pi}{3}$

$= \frac{-\sqrt{3}}{2}$ (1)

b) $\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2$

$= 2$ (1)

c) $\int_1^5 [f(x) + kx]\,dx = \int_1^5 f(x)\,dx + \int_1^5 kx\,dx$

$4 + \left[\frac{kx^2}{2}\right]_1^5 = 28$

$\frac{25k}{2} - \frac{k}{2} = 24$

$24k = 48$

$k = 2$ (3)

d) i) $A = \frac{10}{3}\left(0 + 0 + 4(5 + 15 + 7) + 2(12 + 14)\right)$

$A = 533\frac{1}{3}\ m^2$ (3)

ii) $V = 533\frac{1}{3} \times 5$

$= 2666\frac{2}{3}\ m^3$ (1)

Question 2

a) $A = \int_0^1 x^3\,dx + \frac{(1 \times 1)}{2} + \frac{(2 \times 2)}{2}$

$= \left[\frac{x^4}{4}\right]_0^1 + 2\frac{1}{2}$

$A = 2\frac{3}{4}$ unit^2 (3)

b) $\pi^c = 180°$

i) $\therefore 2x° = \frac{2 \times \pi x}{180}$

$= \frac{\pi x}{90}$ radians (1)

ii) $\int \sin 2x°\,dx = \int \sin \frac{\pi}{90} x\,dx$

$= -\frac{90}{\pi} \cos \frac{\pi}{90} x + c$ (2)

c) i) sim eq $y = x$ $x^2 + y^2 = 2$

$x^2 + x^2 = 2$

$2x^2 = 2$

$x_1 = 1$ (1)

ii) $V = \pi \int_0^1 x^2\,dx + \pi \int_1^{\sqrt{2}} (2 - x^2)\,dx$

$= \pi \left\{ \left[\frac{x^3}{3}\right]_0^1 + \left[2x - \frac{x^3}{3}\right]_1^{\sqrt{2}} \right\}$

$= \pi \left[\frac{1}{3} + \left(2\sqrt{2} - \frac{2\sqrt{2}}{3}\right) - \left(2 - \frac{1}{3}\right) \right]$

$= \pi \left[-\frac{4}{3} + \frac{6\sqrt{2} - 2\sqrt{2}}{3} \right]$

$= \pi \left[\frac{4\sqrt{2} - 4}{3} \right]$ (3)

Question 3

a) $y = a\cos nx + b\sin nx$

$\frac{dy}{dx} = -an\sin nx + bn\cos nx$

$\frac{d^2y}{dx^2} = -an^2\cos nx - bn^2\sin nx$ sub into $\frac{d^2y}{dx^2} + n^2 y = 0$

LHS $= -an^2\cos nx - bn^2\sin nx + n^2(a\cos nx + b\sin nx)$

$= 0$

b) i) $y = x\sqrt{x+3}$

Let $u = x$ $\quad v = \sqrt{x+3} = (x+3)^{1/2}$

$u' = 1$ $\quad v' = \frac{1}{2}(x+3)^{-1/2} = \frac{1}{2\sqrt{x+3}}$

$\therefore \frac{dy}{dx} = \sqrt{x+3} + \frac{x}{2\sqrt{x+3}}$

$= \frac{2(x+3)+x}{2\sqrt{x+3}}$

$= \frac{3x+6}{2\sqrt{x+3}}$

$\frac{dy}{dx} = \frac{3}{2}\left[\frac{x+2}{\sqrt{x+3}}\right]$ (2)

ii) $\therefore \int \frac{x+2}{\sqrt{x+3}}\,dx = \frac{2}{3}x\sqrt{x+3} + c$ (1)

c) i) $P = 2r +$ arc length AB

$= 2r + r\theta$ ——— (1)

since $\frac{1}{2}r^2\theta = 25$

$\theta = \frac{50}{r^2}$ sub into (1)

$\therefore P = 2r + r\left[\frac{50}{r^2}\right]$ (1)

$P = 2r + \frac{50}{r} = 2r + 50r^{-1}$

ii) $\frac{dP}{dr} = 2 - 50r^{-2}$

$\frac{d^2P}{dr^2} = 100r^{-3}$

st pts $2 - \frac{50}{r^2} = 0$

$2r^2 = 50$

$r = \pm 5$ $\quad r > 0 \therefore r = 5$

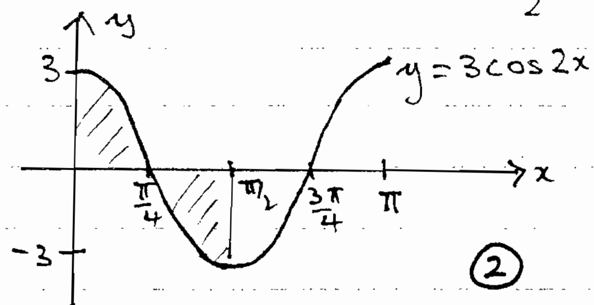
test max/min

if $r = 5$ $\quad \frac{d^2P}{dr^2} > 0 \quad \therefore$ min (3)

$\therefore$ min Perimeter if $r = 5$cm

Question 4

a) i) amplitude $= 3$ $\quad$ period $\frac{2\pi}{2} = \pi$



(2)

$\pi/4$

ii) $A = 2\int_0^{\pi/4} 3\cos 2x\,dx$

$= 6\left[\frac{1}{2}\sin 2x\right]_0^{\pi/4}$

$= 3\left[\sin\frac{\pi}{2} - \sin 0\right]$

$= 3$ unit^2 (3)

b) i) $A = \sqrt{1+3}$ $\quad \therefore A = 2$

$2\left[\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right] = A\sin(x+\theta)$

$\cos\theta = \frac{1}{2}$ $\quad \sin\theta = \frac{\sqrt{3}}{2}$ $\therefore \theta = \frac{\pi}{3}$

$\therefore \sin x + \sqrt{3}\cos x = 2\sin\left(x + \frac{\pi}{3}\right)$ (2)

ii) $2\sin\left(x+\frac{\pi}{3}\right) = \sqrt{2}$

$\sin\left(x+\frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$

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$x + \frac{\pi}{3} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4} \ldots$

$\therefore x = \frac{5\pi}{12}, \frac{23\pi}{12}$ (3)

Question 5

a) $u = \sqrt{x} = x^{1/2}$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$du = \frac{dx}{2\sqrt{x}}$$

$$\therefore dx = 2\sqrt{x}\,du \quad (3)$$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = \int \frac{\cos u}{\sqrt{x}} \cdot 2\sqrt{x}\,du$$

$$= 2\int \cos u\,du$$

$$= 2\sin u + c$$

$$= 2\sin\sqrt{x} + c$$

b) $u = 1 - x$

$$\frac{du}{dx} = -1$$

$$-du = dx$$

$$\int \frac{x}{\sqrt{1-x}}dx = \int \frac{1-u}{\sqrt{u}} \cdot -du$$

$$= -\int (1-u)u^{-1/2}du$$

$$= -\int \left(u^{-1/2} - u^{1/2}\right)du$$

$$= -\left[\frac{u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\right]$$

(4)

$$= -2\sqrt{1-x} + \frac{2}{3}\sqrt{(1-x)^3} + c$$

c) $\cos 2\theta = 2\cos^2\theta - 1$

$$\therefore \int \cos^2 3x\,dx = \frac{1}{2}\int(\cos 6x + 1)dx$$

$$= \frac{1}{2}\left[\frac{1}{6}\sin 6x + x\right] + c$$

Question 6

a) $u = x^2 + 2$ $\qquad x = 1 \rightarrow u = 3$

$\frac{du}{dx} = 2x$ $\qquad x = 0 \rightarrow u = 2$

$$dx = \frac{du}{2x}$$

$$\int_0^1 \frac{x}{(x^2+2)^2}dx = \int_2^3 \frac{x}{u^2} \cdot \frac{du}{2x}$$

$$= \frac{1}{2}\int_2^3 u^{-2}\,du$$

$$= \frac{1}{2}\left[-\frac{1}{u}\right]_2^3$$

$$= \frac{1}{12} \quad (3)$$

b) i) LHS $= \sin(A+B) + \sin(A-B)$

$= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B$

$= 2\sin A.\cos B$

$=$ RHS $\quad (1)$

ii) $\sin 4x.\cos 2x = \frac{1}{2}[\sin 6x + \sin 2x]$

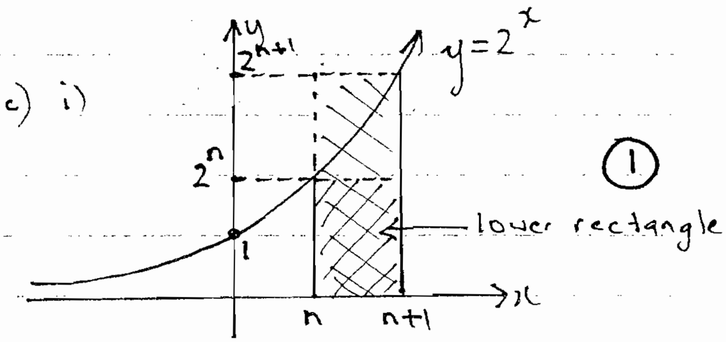
$$\frac{1}{2}\int_0^{\pi/6}(\sin 6x + \sin 2x)dx$$

$$= \frac{1}{2}\left[-\frac{1}{6}\cos 6x - \frac{1}{2}\cos 2x\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left[-\frac{1}{6}\cos\pi - \frac{1}{2}\cos\frac{\pi}{3} - \left(-\frac{1}{6} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{6} - \frac{1}{4} + \frac{1}{6} + \frac{1}{2}\right] = \frac{7}{24} \quad (3)$$

c) i)



(1)

ii) area lower rectangle $< \int_n^{n+1} 2^x dx <$ area upper rectangle

$$2^n \times 1 < \int_n^{n+1} 2^x dx < 2^{n+1} \times 1$$

$$2^n < \int_n^{n+1} 2^x dx < 2.2^n$$