

SYDNEY TECHNICAL HIGH SCHOOL
YEAR 12 HSC ASSESSMENT TASK 2
MARCH 2007
MATHEMATICS

Extension 1

Time Allowed: 70 minutes

Instructions: Attempt all questions

Start each question on a new page

Show all necessary working

The marks for each question are indicated next to the question

Marks may be deducted for careless or badly arranged work

Marks indicated are a guide only and may be varied if necessary

Name: _____ Teacher: _____

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total

QUESTION 1 - (9 marks)

a) You are given $\int_0^a f(x)dx = A$. Evaluate $\int_{-a}^a f(x) dx$ if 2

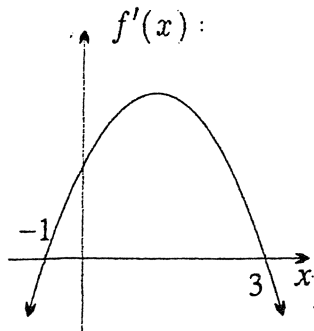
i) $f(x)$ is an even function

ii) $f(x)$ is an odd function

b) Evaluate $\int_0^2 (4-2x)^3 dx$ 2

c) For what values of x is $f(x) = x^5 - 5x^4$ concave down? 2

d) The diagram shows the graph of $f'(x)$ which is the derivative of a certain function $f(x)$ 3

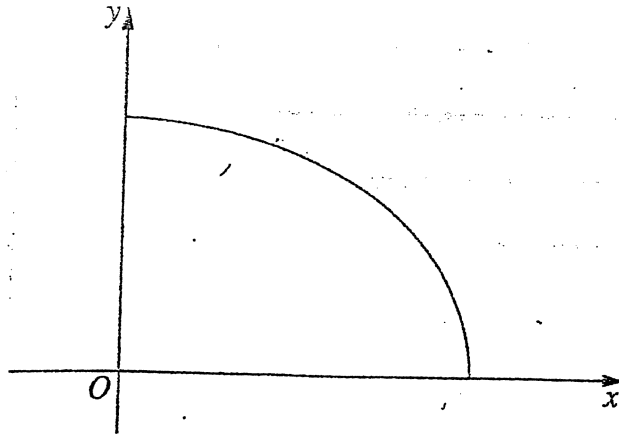


Given that $f(0) = 0$, sketch the graph of $f(x)$

QUESTION 2 - (9 Marks)

a) Find a primitive of $\frac{1}{2x^2}$ 1

b) 4



Part of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is shown above

The following table gives values for the graph

x	0	1	2	3	4
y	3	2.90	2.60	1.98	0

i) Use Simpsons rule and all 5 function values to find an approximation to the area under the curve shown above (2 dec).

ii) If the area of the whole ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

is πab , use this result and your answer above to find an approximate value of π to 2 decimal places.

c) Find $\int \frac{x}{\sqrt{4-x}} dx$ using the substitution $x = 4 - u$ 4

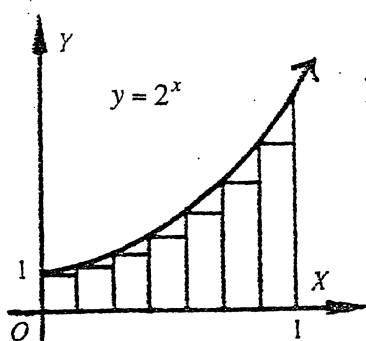
QUESTION 3 - (8 Marks)

a) i) Show that the sum of

3

$$1 + 2^a + 2^{2a} + \dots + 2^{(n-1)a} = \frac{2^{na} - 1}{2^a - 1}$$

ii)



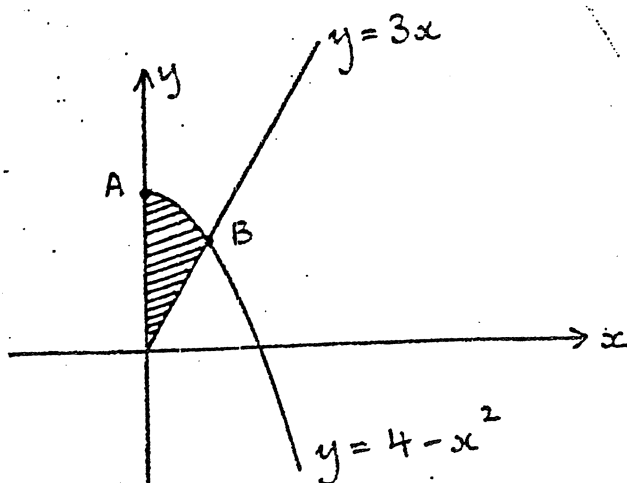
Using 100 inscribed rectangles as shown above, find an approximation for

$$\int_0^1 2^x dx$$

you may use the result in part (i)

b)

5



The sketch above shows $y = 4 - x^2$ and $y = 3x$ for $x \geq 0$

i) Find the co-ordinates of A and B

ii) The shaded area is rotated around the y axis. Find the volume of the solid formed.

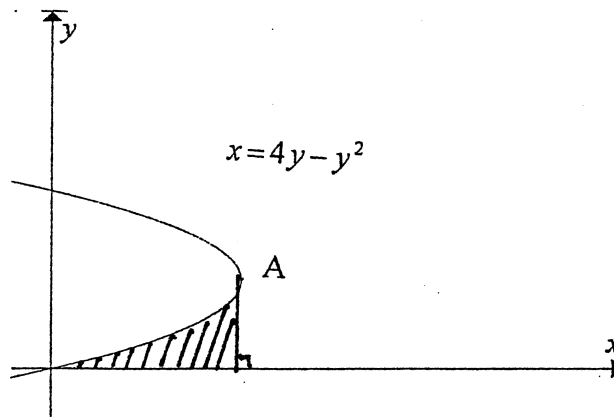
QUESTION 4 - (8 Marks)

- a) Suppose the cubic $f(x) = x^3 + ax^2 + bx + c$ has a relative maximum at $x = \alpha$ and a relative minimum at $x = \beta$. 3

i) Prove that $\alpha + \beta = -\frac{2}{3}a$

ii) Deduce that the point of inflexion occurs at $x = \frac{\alpha + \beta}{2}$

b) 5



- i) Find the co-ordinates of A, the vertex of the parabola.
- ii) By completing the square, make y the subject of $x = 4y - y^2$
- iii) Hence or otherwise find the shaded area

QUESTION 5 - (8 Marks)

- a) The number of unemployed people u at time t was studied over a period of time. At the start of this period, the number of unemployed was 800 000. 2

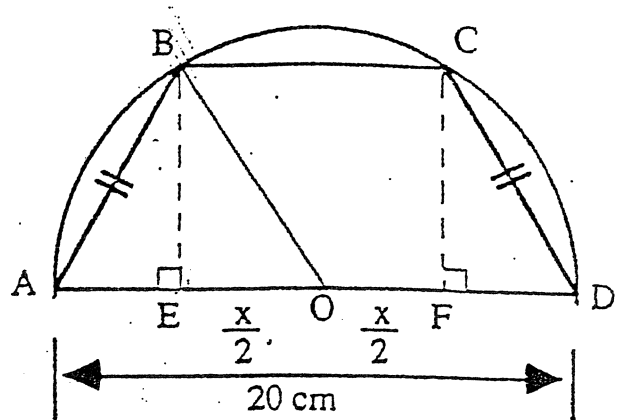
i) Throughout the period, $\frac{du}{dt} < 0$.

What does this say about the number of unemployed during the period?

ii) It is also observed that, throughout the period, $\frac{d^2u}{dt^2} > 0$.

Sketch a graph of u against t .

- b) An isosceles trapezium ABCD is drawn with its vertices on a semicircle centre O and diameter 20cm (see diagram). 6



i) If $EO = OF = \frac{x}{2}$, show that:

$$BE = \frac{1}{2}\sqrt{400 - x^2}$$

- ii) Show that the area ($A \text{ cm}^2$) of the trapezium ABCD is given by:

$$A = \frac{1}{4}(x + 20)\sqrt{400 - x^2}$$

iii) Show that $\frac{dA}{dx} = \frac{1}{4} \left[\frac{400 - 20x - 2x^2}{\sqrt{400 - x^2}} \right]$

- iv) Hence find the length of BC so that the area of trapezium ABCD is a maximum.

QUESTION 6 (8 Marks)

a) i) Solve $\frac{x+1}{(x-1)^2} > 0$

For the curve $y = \frac{x+1}{(x-1)^2}$

- ii) Write down the equations of the asymptotes
- iii) Find the co-ordinates of the stationary point and determine its nature.
- iv) Sketch the curve showing the stationary point, the asymptotes and any intercepts.
- v) Mark on your graph, labelling clearly, the approximate position of any points of inflexion.

QUESTION 1

a) i) 2A

ii) 0

b) $\int_0^2 (4-2x)^3 dx = \left[\frac{(4-2x)^4}{-8} \right]_0^2$

$= 0 - \left(\frac{4^4}{8} \right)$

32

c) concave \downarrow $f''(x) < 0$

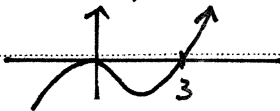
$f(x) = x^5 - 5x^4$

$f'(x) = 5x^4 - 20x^3$

$f''(x) = 20x^3 - 60x^2$

$20x^3 - 60x^2 < 0$

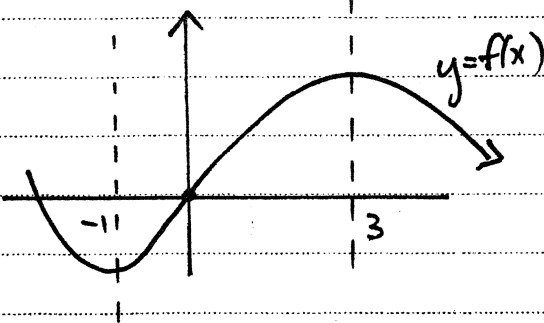
$20x^2(x-3) < 0$



$x < 0$

$0 < x < 3$

d)



QUESTION 2

a) $\frac{1}{2} \int x^{-2} dx = \frac{1}{2} \left[\frac{x^{-1}}{-1} \right] + c$

$= \frac{-1}{2x} + c$

b)

	0	1	2	3	4
y	3	2.9	2.6	1.98	0
F		y_1	y_2	y_3	L

i) $A = \frac{1}{3} [3 + 0 + 4(2.9 + 1.98) + 2(2.6)]$
 $= 9.24$

ii) $\pi ab = 9.24 \times 4$ $a=4$ $b=3$

$12\pi = 9.24 \times 4$

$\pi = 3.08$

c) $x = 4 - u$ $u = 4 - x$

$\frac{dx}{du} = -1$

$dx = -du$

$\int \frac{x}{\sqrt{4-x}} dx = \int \frac{4-u}{\sqrt{u}} \cdot -du$

$= - \int (4-u) u^{-1/2} du$

$= - \int 4u^{-1/2} - u^{1/2} du$

$= - \left[\frac{4u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2} \right] + c$

$= -8\sqrt{u} + \frac{2\sqrt{u^3}}{3} + c$

$= -8\sqrt{4-x} + \frac{2}{3}\sqrt{(4-x)^3} + c$

QUESTION 3

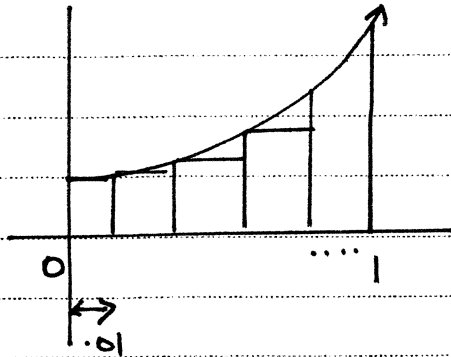
a) $a=1$ $r=2^a$

i) $1 + 2^1 + 2^{2^1} + \dots + 2^{(n-1)^a}$

T_1 T_2 T_3 T_n

$S_n = \frac{1(2^{na} - 1)}{2^a - 1}$ $n=$ required

ii)



$$= \frac{1}{100} [2^0 + 2^{.01} + 2^{.02} + \dots + 2^{0.99}]$$

$$= \frac{1}{100} \left[\frac{2^{100 \times .01} - 1}{2^{.01} - 1} \right]$$

$$= \underline{\underline{1.44}}$$

b) i) A(0, 4) B(1, 3)

$$3x = 4 - x^2$$

$$x^2 + 3x - 4 = 0$$

$$(x + 4)(x - 1) = 0$$

$$x = -4 \quad x = 1$$

ii)

$$V = \pi \int_0^3 \left(\frac{y}{3}\right)^2 dy + \pi \int_3^4 (4 - y) dy$$

$$= \pi \left[\frac{y^3}{27} \right]_0^3 + \pi \left[4y - \frac{y^2}{2} \right]_3^4$$

$$= \pi \left[1 + (16 - 8) - (12 - \frac{9}{2}) \right]$$

$$= \underline{\underline{\frac{3\pi}{2} \text{ unit}^3}}$$

QUESTION 4

a) $f(x) = x^3 + ax^2 + bx + c$

i) max $x = \alpha$

min $x = \beta$

$$f'(x) = 3x^2 + 2ax + b$$

$$f'(\alpha) = 0 \quad 3\alpha^2 + 2a\alpha + b = 0 \quad \text{--- (1)}$$

$$f'(\beta) = 0 \quad 3\beta^2 + 2a\beta + b = 0 \quad \text{--- (2)}$$

① - ②

$$3\alpha^2 - 3\beta^2 + 2a\alpha - 2a\beta = 0$$

$$3(\alpha^2 - \beta^2) + 2a(\alpha - \beta) = 0$$

$$3(\alpha - \beta)(\alpha + \beta) + 2a(\alpha - \beta) = 0$$

$$3(\alpha + \beta) + 2a = 0$$

$$3(\alpha + \beta) = -2a$$

$$\underline{\underline{\alpha + \beta = -\frac{2a}{3}}}$$

ii) Inflection $f''(x) = 0$

$$6x + 2a = 0$$

$$x = -\frac{2a}{6}$$

$$x = -\frac{a}{3}$$

from i) $\alpha + \beta = -\frac{2a}{3}$

$$\frac{3}{2}(\alpha + \beta) = a \text{ sub into *}$$

$$x = -\frac{1}{3} \left(-\frac{3}{2} (\alpha + \beta) \right)$$

$$= \underline{\underline{\frac{\alpha + \beta}{2}}}$$

b) i) A(4, 2)

ii) $x = 4y - y^2$

$$y^2 - 4y + 4 = -x + 4$$

$$(y - 2)^2 = 4 - x$$

$$y-2 = \pm \sqrt{4-x}$$

$$y = 2 \pm \sqrt{4-x}$$

use $y = 2 - \sqrt{4-x}$

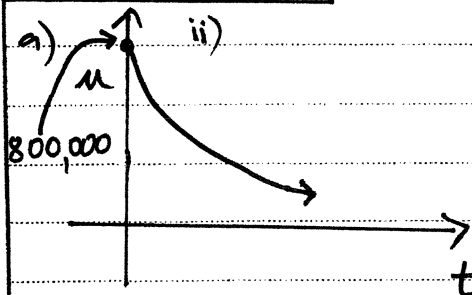
$$A = \int_0^4 2 - (4-x)^{1/2} dx$$

$$= \left[2x + \frac{2(4-x)^{3/2}}{3} \right]_0^4$$

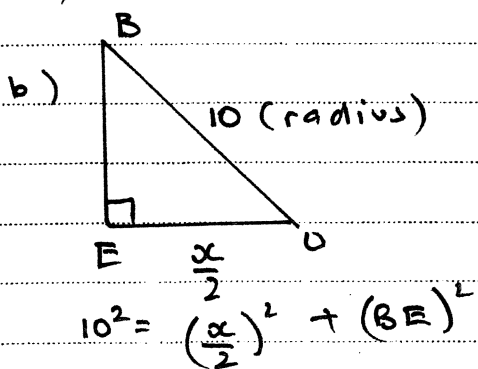
$$= 8 - \left(0 + \frac{16}{3} \right)$$

$$= \underline{\underline{\frac{8}{3} \text{ unit}^2}}$$

QUESTION 5



i) no. unemployed decreasing



$$(BE)^2 = 100 - \frac{x^2}{4}$$

$$BE = \sqrt{\frac{400 - x^2}{4}}$$

$$BE = \underline{\underline{\frac{1}{2} \sqrt{400 - x^2}}}$$

ii) $A = \frac{1}{2} h (a+b)$
Trap

$$= \frac{1}{2} \left(\frac{1}{2} \sqrt{400-x^2} \right) (20+x)$$

$$= \frac{1}{4} (x+20) (\sqrt{400-x^2})$$

iii) $u = \frac{1}{4} (x+20)$ $v = (400-x^2)^{1/2}$

$$u' = \frac{1}{4}$$

$$v' = \frac{1}{2} \cdot -2x (400-x^2)^{-1/2}$$

$$v' = \frac{-x}{\sqrt{400-x^2}}$$

$$\frac{dA}{dx} = \frac{\sqrt{400-x^2}}{4} - \frac{x(x+20)}{4\sqrt{400-x^2}}$$

$$= \frac{400-x^2 - x(x+20)}{4\sqrt{400-x^2}}$$

$$= \frac{400 - x^2 - x^2 - 20x}{4\sqrt{400-x^2}}$$

$$= \frac{1}{4} \left[\frac{400 - 2x^2 - 20x}{\sqrt{400-x^2}} \right]$$

iv) st pt $A' = 0$

$$400 - 2x^2 - 20x = 0$$

$$200 - x^2 - 10x = 0$$

$$(x-10)(x+20) = 0$$

$$x = 10$$

x	10^-	10	10^+
A'	$+$	0	$-$

\therefore Max $x = 10$

$$\underline{\underline{BC = 10 \text{ cm}}}$$

QUESTION 6

i) $\frac{(x+1) > 0}{(x-1)^2}$ $x+1 > 0$
 $x > -1$
but $x \neq 1$

ii) asymptotes $x=1$
 $x \rightarrow \infty$ $y = 0^+$
 (from +ve side)
 $x \rightarrow -\infty$ $y = 0^-$
 (from -ve side)

iii) $u = x+1$ $v = (x-1)^2$
 $u' = 1$ $v' = 2(x-1)$

$$y' = \frac{(x-1)^2 - 2(x+1)(x-1)}{(x-1)^4}$$

st pt

$$(x-1)^2 - 2(x+1)(x-1) = 0$$

$$(x-1)[(x-1) - 2(x+1)] = 0$$

$x=1$ $x-1-2x-2=0$

$$-x-3=0$$

$x=-3$

$x \neq 1 \therefore$

st pt $x = -3$

x	-4	-3	-2
y'	-	0	+

- \ / +
 - 0 +

min

at $(-3, -\frac{1}{8})$

