Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC COURSE

Extension 1 Mathematics

HSC Task 2

March 2009

<u>TIME ALLOWED</u>: **70 minutes**

Instructions:

- Write your name and class at the top of this page, and on all your answer sheets.
- Hand in your answers attached to the rear of this question sheet.
- All necessary working must be shown. <u>Marks may not be awarded for careless or badly arranged work.</u>
- Marks indicated are a guide only and may be varied at the time of marking

(FOR MARKERS USE ONLY)

1	2	3	4	5	6	TOTAL
/11	/11	/11	/11	/11	/11	/66

QUESTION 1 (11 Marks):

Marks

(a) (i) Find the points of intersection of the curves $y = x^2$ and $y = 3x - 2$ **1**

(ii) Calculate the area between the two curves **2**

(b) Given that $\frac{d^2y}{dx^2} = 12x\text{-}2$, and that when $x = 0$, $\frac{dy}{dx} = 0$ and $y = 4$, **2**

find y in terms of x.

(c) The section of the curve $y = \sqrt{4 - x^2}$ between x=1 and x=2 is rotated about the x-axis.

(i) describe or draw a sketch of the solid so formed **1**

(ii) Find the volume of the solid, leaving your answer in terms of π. **3**

(d) By using the substitution $u = 1 - x^2$, or otherwise, find **2**

$$\int x\sqrt{1 - x^2}\, dx$$

QUESTION 2 (11 Marks):

(a) Show that if $y = (1 - x)(x + 1)^3$ then $\frac{dy}{dx} = 2(x + 1)^2(1 - 2x)$ **2**

(b) You are given that $\frac{d^2y}{dx^2} = -12x(x + 1)$. **4**
Find all stationary points on the curve and their nature.

(c) Find all points of inflexion. **2**

(d) Sketch the curve, showing all major features **3**

QUESTION 3 (11 marks):

Marks

(a) Find

(i) $\int \frac{x^3+1}{x^2}\, dx$ **2**

(ii) $\int_0^4 \frac{dx}{\sqrt{x}}$ **2**

(b) A piece of wire 28 cm long is cut and then bent to form a rectangle and a square.

(i) If the width of the rectangle is x cm and the length is 3 times its width, show that the sum of the areas of the rectangle and square is given by **2**

$$A = 7x^2 - 28x + 49$$

(ii) If A is to be a minimum, find the area of the square. (Justify that your answer is a minimum.) **3**

(c) Differentiate $\sqrt{1-2x}$ and hence, or otherwise, find $\int \frac{1}{\sqrt{1-2x}}\, dx$ **2**

QUESTION 4 (11 Marks):

(a) The table of values below describes the function $y = f(x)$ **2**

x	0	0.5	1	1.5	2
$f(x)$	1	1.649	2.718	4.482	7.389

Using Simpson's Rule with 5 function values, approximate $\int f(x)dx$
Give your answer correct to 2 decimal places.

(b) Solve $2\sin 2x = \sqrt{3}$ for $0 \leq x \leq \pi$ **4**

(c) Using the substitution $u=1-x$, or otherwise, evaluate **5**

$$\int_0^1 x\sqrt{1-x}\; dx$$

QUESTION 5 (11 Marks):

Marks

(a) (i) Express $\cos\theta - \sin\theta$ in the form $A\cos(\theta+\alpha)$ where $A>0$ and $0 \le \alpha \le \frac{\pi}{2}$ **2**

(ii) Hence, or otherwise, solve **2**

$$\cos\theta - \sin\theta = 1 \quad \text{for } 0 \le \theta \le 2\pi$$

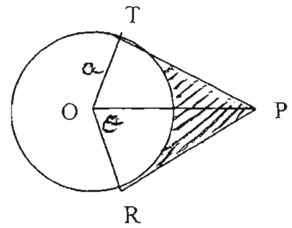
(iii) If the domain of part (ii) above was changed to $-\pi \le \theta \le \pi$ **1**

give the new solutions to $\cos\theta - \sin\theta = 1$

(b) Two tangents are drawn to a circle, centre O, of radius a units from a point P.

P is $2a$ units from the centre of the circle.

THE DIAGRAM IS NOT TO SCALE



(i) Find the area of ΔOTP **2**

(ii) Find the size of the angle TOR **1**

(iii) Find the area of the shaded section. **3**

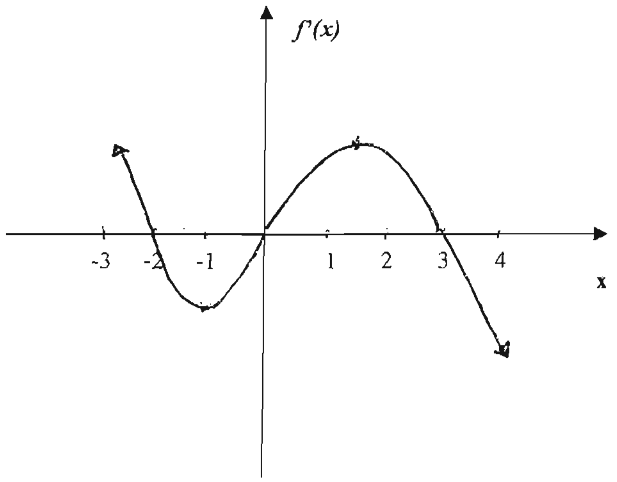
QUESTION 6 (11 Marks):

Marks

(a) For a certain curve $y = f(x)$, the graph of $y = f'(x)$ is sketched below. **4**

You are also given that $f(3)=3$, $f(0) = -3$ and $f(-2) = 1$.

Sketch a possible graph of $y=f(x)$ over $-3 \le x \le 4$

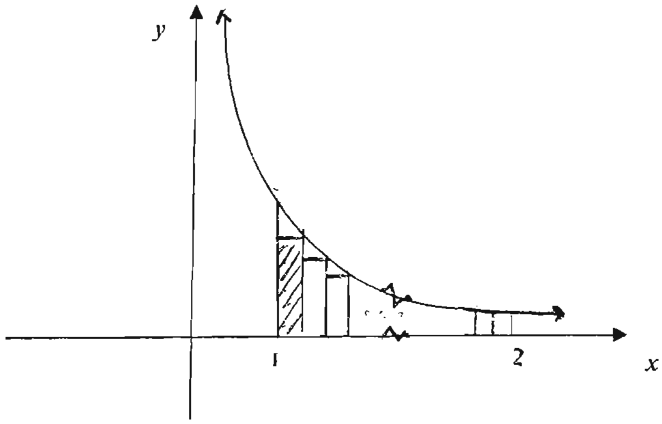


QUESTION 6 continues over the page.....

QUESTION 6 continued......

Marks

(b) The curve $y = \frac{1}{x^2}$ is shown below



The area below the curve and between the ordinates $x=1$ and $x=2$ has been divided into n rectangles of equal width, as shown.

(i) Find the value of $\int_1^2 \frac{dx}{x^2}$ **1**

(ii) Show that the area of the shaded rectangle is $\frac{n}{(n+1)^2}$ **3**

(iii) Find an expression for the area of the rectangle directly to the right of the shaded one. **1**

(iv) You are given that the area of the last rectangle on the right is $\frac{n}{(2n)^2}$ **2**

Show that

$$\lim_{n \to \infty} n\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots\dots\dots + \frac{1}{(2n)^2}\right) = \frac{1}{2}$$

END OF EXAMINATION PAPER

SOLUTIONS

QUESTION 1

(a) (i) $y = x^2$

$y = 3x - 2$

$x^2 = 3x - 2$

$\therefore (x-2)(x-1) = 0$

$\therefore \begin{cases} x = 2 \\ y = 4 \end{cases}$ or/ $\begin{cases} x = 1 \\ y = 1 \end{cases}$ (1)

(ii) $A = \int_1^2 3x - 2 - x^2 \, dx \leftarrow$ (1)

$= \left[\tfrac{3}{2}x^2 - 2x - \tfrac{1}{3}x^3\right]_1^2$

$= \left(6 - 4 - \tfrac{8}{3}\right) - \left(\tfrac{3}{2} - 2 - \tfrac{1}{3}\right)$

$= \tfrac{1}{6} \, u^2 \leftarrow$ (1)

OR/ 1 MARK for each area

(b) $\dfrac{dy}{dx} = 6x^2 - 2x + k$

At $x = 0$, $\tfrac{dy}{dx} = 0$

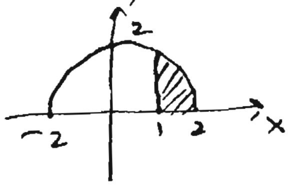
$\therefore k = 0$

$\therefore \tfrac{dy}{dx} = 6x^2 - 2x$ (1)

$y = 2x^3 - x^2 + c$

At $x = 0$, $y = 4$ $\therefore c = 4$

$\therefore y = 2x^3 - x^2 + 4$ (1)

(c) 

BOWL/ CONTACT LENS. (1)

(ii) $\text{VOL} = \pi \int_1^2 y^2 \, dx$ (1)

$= \pi \int_1^2 (4 - x^2) \, dx$ (1)

$= \pi \left[4x - \tfrac{1}{3}x^3\right]_1^2$

$= \tfrac{5\pi}{3}$ cu units (1)

(d) Let $u = 1 - x^2 \Rightarrow \dfrac{du}{dx} = -2x$

$dx = -\dfrac{du}{2x}$

$\therefore \int x\sqrt{1-x^2} \, dx$

$= \int x\sqrt{u} \left(-\tfrac{du}{2x}\right)$ (1)

$= -\tfrac{1}{2} \int \sqrt{u} \, du$

$= -\tfrac{1}{2} \cdot \tfrac{2}{3} u^{\frac{3}{2}} + k$

$= -\tfrac{1}{3} (1 - x^2)^{3/2} + k$ (1)

OR/ $\int x\sqrt{1-x^2} \, dx$

$= \tfrac{2}{3}(1 - x^2)^{3/2} \cdot \tfrac{1}{2} + k$

$= -\tfrac{1}{3}(1 - x^2)^{3/2} + k.$ (2)

[guess and correct method]

QUESTION 2:

(a) $y = (1-x)(1+x)^3$

$$\frac{dy}{dx} = (1+x)^3(-1) + (1-x)3(1+x)^2 \leftarrow 1$$

$$= (1+x)^2\left[-1-x+3-3x\right]$$

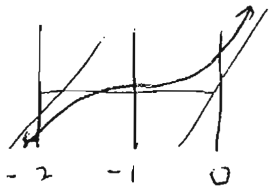
$$= (1+x)^2(2-4x)$$

$$= 2(1+x)^2(1-2x) \leftarrow 1$$

(b) At S.P.'s $\frac{dy}{dx} = 0$

$1 \rightarrow \begin{cases} x=-1 \\ y=0 \\ y''=0 \end{cases}$ or $\begin{cases} x = \frac{1}{2} \leftarrow 1 \\ y = \frac{27}{16}. \\ y''<0 \Rightarrow \text{max.} \quad 1 \\ \text{at } \left(\frac{1}{2}, \frac{27}{16}\right) \end{cases}$

$\frac{dy}{dx}$



I.P. at $(-1, 0)$ 1

(c) At I.P.'s $\frac{d^2y}{dx^2} = 0$

$\therefore -12x(x+1) = 0$

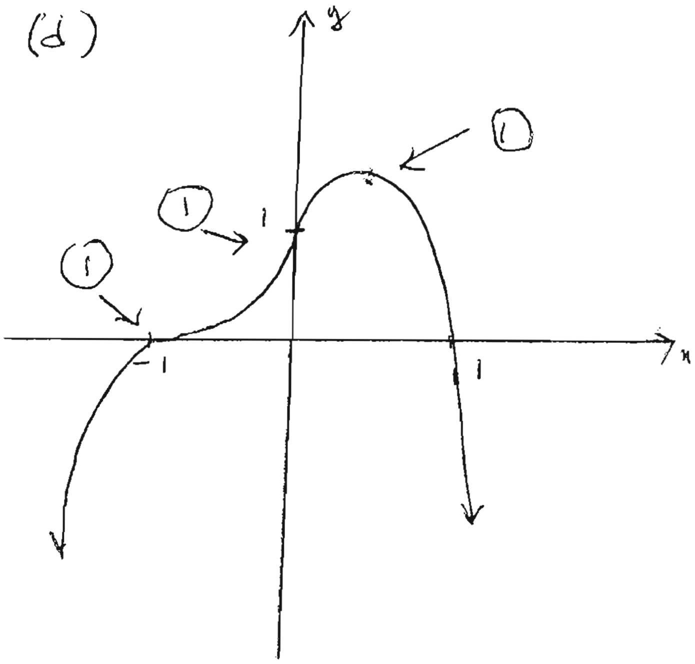
$1 \quad \therefore \begin{cases} x=0 \\ y=1 \end{cases}$ or $x=-1$ found

y''	$-\frac{1}{2}$	0	$\frac{1}{2}$
	+	0	−

$\leftarrow 1$

$\therefore$ I.P.'s at $(0,1)$ and $(-1,0)$

(d)



QUESTION 3:

(a) (i) $\int x + \frac{1}{x^2}\, dx = \frac{1}{2}x^2 + -\frac{1}{x} + k.$ Penalty of 1 for no constant here ONLY

2 $\quad \frac{1}{2}x^2 - \frac{1}{x} + k$

(ii) $\left[2x^{\frac{1}{2}}\right]_0^4 = 4$

1 1

(b) (i) $A_R = 3x^2 \quad A_S = \left(\frac{28 - 8x}{4}\right)^2$ ← 1

1

$\therefore$

$A = 3x^2 + 49 - 28x + 4x^2$

$= 7x^2 - 28x + 49.$

(ii) $\frac{dA}{dx} = 14x - 28 \qquad \frac{d^2A}{dx^2} = 14 > 0 \Rightarrow$ MIN. 1

$\therefore x = 2$ 1

$\therefore y = 3.$

$\therefore$ Area of square $= 9$ un^2. 1

(c) $\frac{d}{dx}(1-2x)^{\frac{1}{2}} = \frac{1}{2}(-2)(1-2x)^{-\frac{1}{2}}.$

$= -\frac{1}{\sqrt{1-2x}}$ 1

$\therefore \int \frac{dx}{\sqrt{1-2x}} = -\sqrt{1-2x} + k$ 1 for minus

No penalty for no k.

QUESTION 4:

(a) $A_1 = \frac{1}{3} \cdot 0.5 \left[1 + 6.596 + 2.718\right]$ ← 1 (1.719)

$A_2 = \frac{1}{3} \cdot 0.5 \left[2.718 + 17.928 + 7.389\right]$ ← 1 (4.6725)

$\therefore \int f(x)\,dx \approx 6.39$

[NOTE: There are many ways to do this!]

(b) $\sin 2x = \sqrt{3}/2$ ← 1 $\quad 0 \leq 2x \leq 2\pi$.

$\therefore 2x = \pi/3,\ 2\pi/3,$ ← 1 or equivalent

$\therefore x = \pi/6,\ \pi/3$ ← 2

(c) $u = 1 - x \Rightarrow du/dx = -1$ $\quad \left[\begin{matrix} x = 0 & u = 1 \\ x = 1 & u = 0 \end{matrix}\right\}$ ← 1

$\therefore dx = -du$ ← 1

$\therefore \int_0^1 x\sqrt{1-x}\,dx = -\int_1^0 (1-u)\sqrt{u}\,du$ ← 1

$= \int_1^0 -\sqrt{u} + u\sqrt{u}\,du$

$= -\frac{2}{3}u^{3/2}\Big]_1^0 + \frac{2}{5}u^{5/2}\Big]_1^0$

$= \frac{2}{3} + \left(-\frac{2}{5}\right)$ ← 2

$= \frac{4}{15}$

OR//

$u = 1 - x \Rightarrow dx = -du$

$\therefore \int x\sqrt{1-x} = \int (1-u)\sqrt{u}\,(-du)$ ← 1

$= u\sqrt{u} - \sqrt{u} + k$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + k$ ← 1

$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2}$ ← 1

$\therefore$ Def. INT $= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2}\Big]_0^1$ ← 1

$= 0 - \frac{2}{5} + \frac{2}{3}$

$= \frac{4}{15}$ ← 1

Question 5:

a(i) $\cos\theta - \sin\theta = \sqrt{2}\left(\cos\theta \tfrac{1}{\sqrt{2}} - \sin\theta \tfrac{1}{\sqrt{2}}\right)$

$= \sqrt{2}\cos(\theta + \alpha)$

where $\cos\alpha = 1/\sqrt{2}$.

$\therefore \alpha = \pi/4$.

$\therefore \cos\theta - \sin\theta = \sqrt{2}\cos(\theta + \pi/4)$

① ①

(ii) $\cos\theta - \sin\theta = 1 \Rightarrow \sqrt{2}\cos(\theta + \pi/4) = 1$

$\therefore \cos(\theta + \pi/4) = 1/\sqrt{2}$

$\therefore \theta + \pi/4 = \pi/4$ or $7\pi/4$

$\therefore \theta = 0$ or $\theta = 3\pi/2$ ← ① for each.

(iii) $\theta = 0$ or $\theta = -\pi/2$ ①

(b) (i) Since $\angle OTP = 90°$, $TP^2 = 4a^2 - a^2$

$TP = a\sqrt{3}$ ← ①

$\therefore$ Area $\triangle OTP = \tfrac{1}{2}.a.a\sqrt{3}$

$= \dfrac{a^2\sqrt{3}}{2}$ ← ①

(ii) $\cos\theta = a/2a = 1/2$

$\therefore \theta = \pi/3$.

$\therefore \angle TOR = 2\pi/3$. ← ①

(iii) Area $\triangle = 2 \times \dfrac{a^2\sqrt{3}}{2}$ ← ①

$= a^2\sqrt{3}$.

Area sector $OTR = \tfrac{1}{2}r^2\theta$

$= \tfrac{1}{2}(a^2) . 2\pi/3$

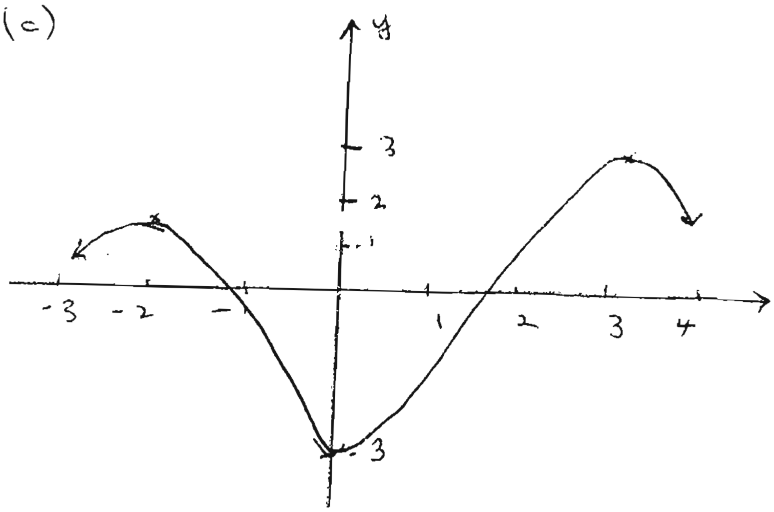
$= \dfrac{\pi a^2}{3}$ ← ①

$\therefore$ Area shaded $= a^2\left(\sqrt{3} - \pi/3\right)$ ← ①

$= \dfrac{a^2}{3}\left(3\sqrt{3} - \pi\right)$

QUESTION 6:

(a)



3 1 each for each of the T.P's

1 for showing the points $(-2,1)$ $(0,-3)$ $(3,3)$

(b)

(i) $\displaystyle\int_1^2 \frac{dx}{x^2} = -x^{-1}\Big]_1^2$

$= \frac{1}{2}$ 1

(ii) width $= \frac{1}{n}$, height is $\dfrac{1}{(1+\frac{1}{n})^2}$ $\therefore$ Area $= \frac{1}{n}\left[\dfrac{1}{(1+\frac{1}{n})^2}\right]$

↑1 ↑1

$= \frac{1}{n} \cdot \dfrac{n^2}{(n+1)^2}$

$= n/(n+1)^2$ 1

(iii) $A_2 = \dfrac{n}{(n+2)^2}$ 1

(iv) Sum of areas of rectangles

$= \dfrac{n}{(n+1)^2} + \dfrac{n}{(n+2)^2} + \cdots + \dfrac{n}{(2n^2)}$

$= n\left[\dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \cdots + \dfrac{1}{(2n)^2}\right]$ 1

The exact area is the limit of this sum as $n \to \infty$

and from part (i) this exact area is $\frac{1}{2}$ 1

$\therefore \displaystyle\lim_{n\to\infty} n\left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}\right] = \frac{1}{2}$