File

Name:_______________________________ Maths Class:__________

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC COURSE

Extension 1 Mathematics

HSC Task 2

March 2010

Time Allowed: **70 minutes**

Instructions:

- Write your name and class at the top of this page, and on all your answer sheets.
- Hand in your answers attached to the rear of this question sheet.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated are a guide only and may be varied at the time of marking

(For Markers Use Only)

1	2	3	4	5	6	Total
/9	/9	/11	/9	/11	/10	/59

QUESTION 1 **(9 marks)**

a) i) Sketch $y = |1 - 2x|$, for $-1 \le x \le 2$ (2)

ii) Hence, evaluate $\int_{-1}^{2} |1 - 2x| dx$ (2)

b) Sketch a continuous curve $y = f(x)$, in the domain $-4 \le x \le 4$, that satisfies all of the following conditions:

$f(x)$ is odd

$f(3) = 0$

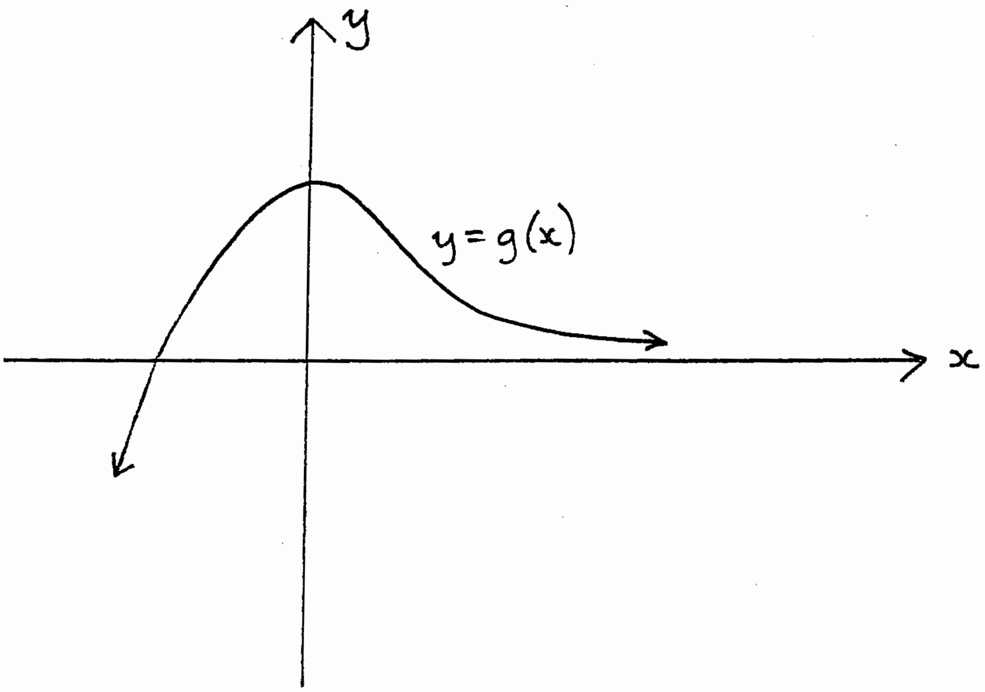
$f'(1) = 0$ (5)

$f'(x) > 0 \; for \; x > 1$

$f'(x) < 0 \; for \; 0 \le x < 1$

QUESTION 2 **(9 Marks)** **(Start a new page)**

a) The function $y = g(x)$ has been sketched below.



Sketch $y = g'(x)$, the derivative function. (2)

b) Using a suitable substitution or otherwise find

$$\int x\sqrt{4-x^2}\,dx \qquad (3)$$

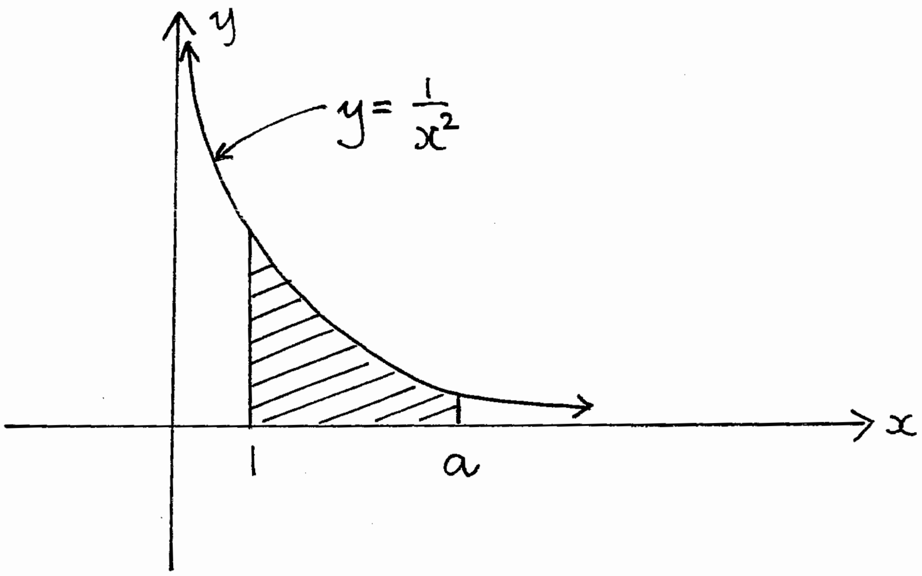
c) Show that $\displaystyle\int_0^{1/4} \frac{3x}{(1+4x)^3}\,dx = \frac{3}{128}$ (4)

Use the substitution $u = 1 + 4x$

QUESTION 3 (11 Marks) (Start a new page)

a) The shaded area below is $\frac{2}{3}$ unit^2. (3)

Find the value of a



b) i) Show that the function $y = \frac{2x^2}{x^2+1}$ has one stationary point and determine its nature. (3)

ii) Find a horizontal asymptote for this function. (1)

iii) Sketch the function showing the stationary point and any asymptotes. (label your sketch clearly) (2)

iv) Without further calculations, indicate with a cross on your sketch, any point(s) of inflexion. (2)

QUESTION 4 (9 Marks) (Start a new page)

a) Prove by mathematical induction that (5)

$3^n + 7^{n+1}$ is divisible by 4 for all positive integers n

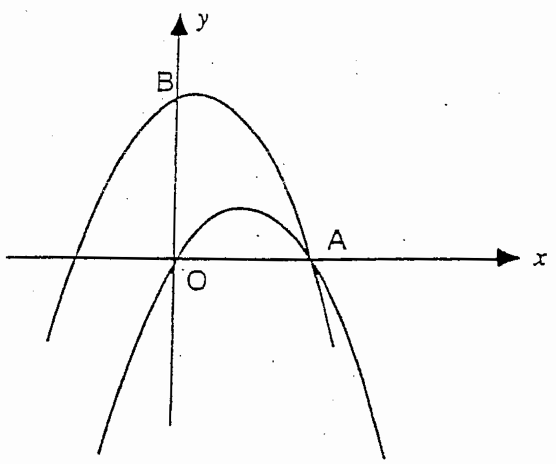
b) An ellipse has the equation $x^2 + 8y^2 = 16$. (The ellipse has its centre at (0,0))

i) Find where the ellipse cuts the y axis. (1)

ii) If the ellipse is rotated around the y axis find the volume of the solid formed. (in exact form) (3)

QUESTION 5 (11 Marks) (Start a new page)

a)



The sketch shows the parabolas
$y = x(3 - x)$ and
$y = (3 - x)(2 + x)$

(i) What are the co-ordinates of A and B? (2)

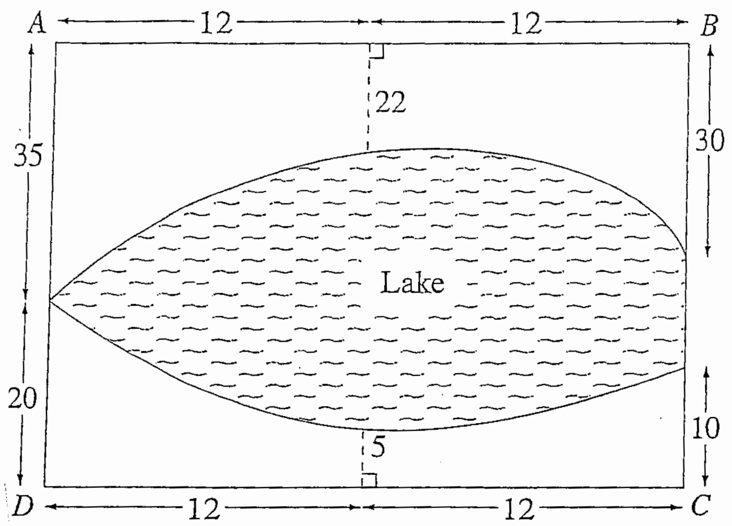
(ii) Prove that the area of the region bounded by OB and the arcs OA and AB is equal to that of ΔOAB (4)

b) i) Sketch $y = x^2$ and $y = 4x - x^2$ on the same axes and clearly indicate the points of intersection. (2)

ii) Find the volume of the hollow cup generated when the area enclosed between the curves $y = x^2$ and $4x - x^2$ is rotated about the x-axis. (in exact form) (3)

QUESTION 6 (10 Marks) (Start a new page)

a) There is a lake inside the rectangular grass picnic area $ABCD$, as shown in the diagram (2)

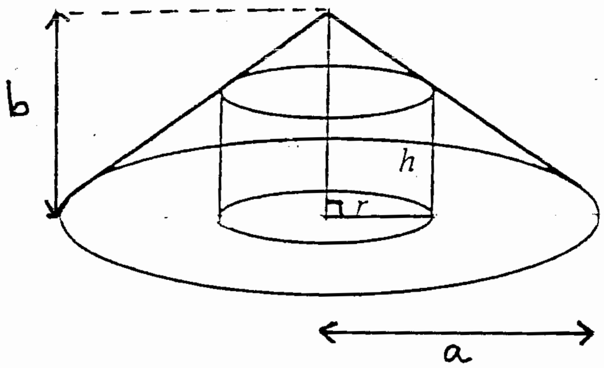


NOT TO SCALE

All measurements are in metres

Use Simpson's Rule to find the approximate area of the lake's surface.

b)



A variable cylinder, radius r and height h, is inscribed in a fixed cone, radius a and height b. (Note: a and b are constants)

i) Prove that $h = \frac{b\,(a-r)}{a}$ (2)

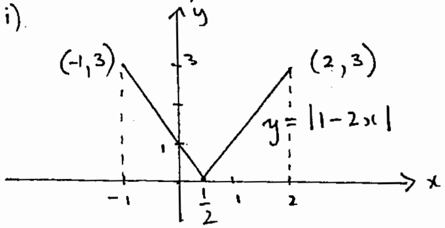
ii) Express the volume of the cylinder as a function of r (1)

iii) Find the maximum volume of the cylinder in terms of a and b (4)

iv) Prove that the cylinder with maximum volume is $\frac{4}{9}$ that of the cone (1)

Question 1

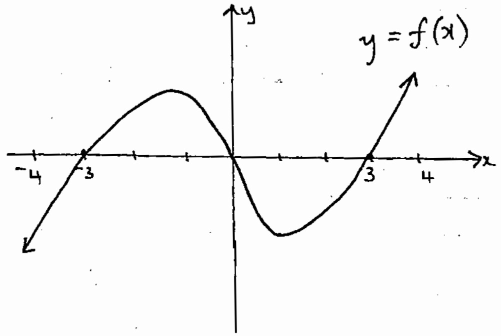
a) i)



ii) $\int_{-1}^{2} |1-2x|\,dx = \dfrac{2\left(1\frac{1}{2}\times 3\right)}{2}$

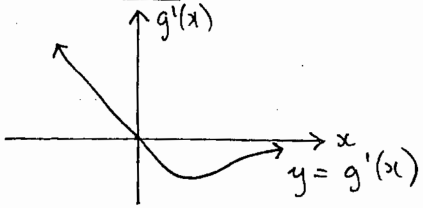
$= 4.5$

b)



Question 2

a)



b) $u = 4 - x^2$

$\dfrac{du}{dx} = -2x$

$du = -2x\,dx$

$\dfrac{du}{-2x} = dx$

$\therefore \int x\sqrt{4-x^2}\,dx = \int x \cdot u^{1/2} \cdot \dfrac{du}{-2x}$

$= -\frac{1}{2}\int u^{1/2}\,du$

$= -\frac{1}{2}\left[\dfrac{2u^{3/2}}{3}\right] + c$

$= -\frac{1}{3}\left(4-x^2\right)^{3/2} + c$

c) $u = 1 + 4x \Rightarrow \dfrac{u-1}{4} = x$

$\dfrac{du}{dx} = 4 \quad \therefore \dfrac{du}{4} = dx$

$x = \frac{1}{4} \quad u = 2$

$x = 0 \quad u = 1$

$\therefore \int_0^{1/4} \dfrac{3x}{(1+4x)^3}\,dx = \int_1^2 3\dfrac{(u-1)}{4} \cdot \dfrac{1}{u^3} \cdot \dfrac{du}{4}$

$= \dfrac{3}{16}\int_1^2 \left(\dfrac{u-1}{u^3}\right) du$

$= \dfrac{3}{16}\int_1^2 u^{-2} - u^{-3}\,du$

$= \dfrac{3}{16}\left[-u^{-1} + \dfrac{u^{-2}}{2}\right]_1^2$

$= \dfrac{3}{16}\left[-\dfrac{1}{u} + \dfrac{1}{2u^2}\right]_1^2$

$= \dfrac{3}{16}\left[\left(-\dfrac{1}{2}+\dfrac{1}{8}\right) - \left(-1+\dfrac{1}{2}\right)\right]$

$= \dfrac{3}{16}\left(-\dfrac{3}{8} + \dfrac{1}{2}\right)$

$= \dfrac{3}{128}$

Question 3

a) $\int_1^a \dfrac{1}{x^2}\,dx = \dfrac{2}{3}$

$\int_1^a x^{-2}\,dx = \dfrac{2}{3}$

$\left[-x^{-1}\right]_1^a = \dfrac{2}{3}$

$\left[-\dfrac{1}{x}\right]_1^a = \dfrac{2}{3}$

$-\dfrac{1}{a} + 1 = \dfrac{2}{3}$

$\dfrac{1}{3} = \dfrac{1}{a}$

$\therefore a = 3$

b) i) $u = 2x^2 \qquad v = x^2+1$

$u' = 4x \qquad v' = 2x$

$\dfrac{dy}{dx} = \dfrac{4x(x^2+1) - 2x \cdot 2x^2}{(x^2+1)^2}$

$= \dfrac{4x^3 + 4x - 4x^3}{(x^2+1)^2}$

$\dfrac{dy}{dx} = \dfrac{4x}{(x^2+1)^2}$

st pt $\dfrac{dy}{dx} = 0 \qquad 4x = 0$

$\therefore x = 0$

at $(0,0)$ test max/min

x	-1	0	1
dy/dx	$-$	0	$+$

\ min /

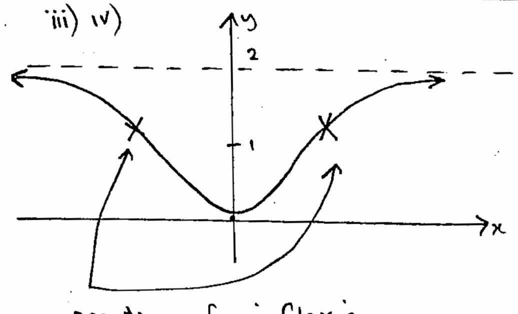
$\therefore (0,0)$ is a min turning pt.

ii) $\lim_{x\to\infty} \dfrac{2x^2}{x^2+1}$

$\lim_{x\to\infty} \dfrac{x^2(2)}{x^2\left(1+\frac{1}{x^2}\right)} = 2$

$\therefore y = 2$ horizontal asymptote

iii) iv)



points of inflexion

Question 4

a) "$3^n + 7^{n+1}$ div by 4 positive n"

Step ① show true for $n = 1$

$3^1 + 7^2 = 52$ div by 4

Step ② assume true for some +ve integer k

$* \quad 3^k + 7^{k+1} = 4M$ (M is an integer)

Step ③ Prove true for $n = k+1$

$3^{k+1} + 7^{k+2} = 3 \cdot 3^k + 7^k \cdot 7^2$

(from $*$) $= 3\left(4M - 7^k \cdot 7^1\right) + 49 \cdot 7^k$

$= 12M - 21 \cdot 7^k + 49 \cdot 7^k$

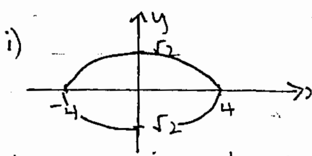
$= 12M + 28 \cdot 7^k$

$= 4\left(3M + 7 \cdot 7^k\right)$

Step ④ Since shown true for $n = 1$ and if assumed true for $n = k$ we have shown true for $n = k+1$ $\therefore$ true for all positive integers.

b) $x^2 + 8y^2 = 16$

i)



$8y^2 = 16$

$y^2 = 2$

$y = \pm\sqrt{2}$

cut y axis at $y = \pm\sqrt{2}$

$$V_y = 2\pi\int_0^{\sqrt2}(16-8y^2)\,dy$$
$$= 2\pi\left[16y-\frac{8y^3}{3}\right]_0^{\sqrt2}$$
$$= 2\pi\left[16\sqrt2-\frac{8}{3}.2\sqrt2\right]$$
$$= 2\pi\left[16\sqrt2-\frac{16\sqrt2}{3}\right]$$
$$V=\frac{64\pi\sqrt2}{3}\text{ unit}^3$$

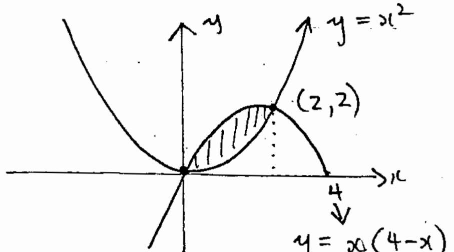
Question 5

a) i) $A(3,0)$ $B(0,6)$

ii) $\Delta OAB = \frac{3\times6}{2} = 9\text{ unit}^2$

$$A_x = \int_0^3 (3-x)(2+x) - x(3-x)\,dx$$
$$= \int_0^3 (6+x-x^2-3x+x^2)\,dx$$
$$= \int_0^3 (6-2x)\,dx$$
$$= \left[6x-x^2\right]_0^3$$
$$= 18-9$$
$$= 9\text{ unit}^2 \text{ equal to } \Delta OAB$$

b)



$$V_x = \pi\int_0^2 (4x-x^2)^2-(x^2)^2\,dx$$
$$= \pi\int_0^2 (16x^2-8x^3+\cancel{x^4}-\cancel{x^4})\,dx$$
$$= \pi\int_0^2 (16x^2-8x^3)\,dx$$
$$= \pi\left[\frac{16x^3}{3}-2x^4\right]_0^2$$
$$= \pi\left[\frac{128}{3}-32\right]$$
$$= \frac{32\pi}{3}\text{ units}^3$$

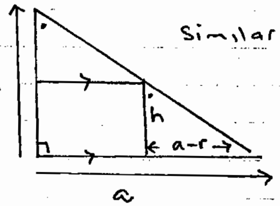
Question 6

x	0	12	24
$f(x)$	0	28	15

a) $A_s = \frac{12}{3}\left[0+15+4(28)\right]$

$= 508\text{ m}^2$

b) i)



similar Δ's

$$\frac{h}{b} = \frac{a-r}{a}$$
$$\therefore h = \frac{b(a-r)}{a}$$

ii) $V_{cylinder} = \pi r^2 h$
$$= \pi r^2 b.\frac{(a-r)}{a}$$
$$\therefore V = \pi r^2 b - \frac{\pi r^3 b}{a}$$

iii) $$\frac{dV}{dr} = 2\pi rb - \frac{3\pi r^2 b}{a}$$
$$\frac{d^2V}{dr^2} = 2\pi b - \frac{6\pi rb}{a}$$

st pt $\frac{dV}{dr}=0$
$$2\cancel{\pi}rb = \frac{3\cancel{\pi}r^2b}{a}$$
$$2abr = 3r^2b$$
$$2abr-3r^2b=0$$
$$br(2a-3r)=0$$

$\therefore r=0$ (no result) since $r>0$; $2a=3r$, $r=\frac{2a}{3}$

test max/min using $\frac{d^2V}{dr^2}$

and $r=\frac{2a}{3}$ $$\frac{d^2V}{dr^2} = 2\pi b - \frac{6\pi b}{\cancel{a}}.\frac{2\cancel{a}}{3}$$
$$= 2\pi b - 4\pi b$$
$$= -2\pi b$$

$\therefore \frac{d^2V}{dr^2}<0$ max volume $r=\frac{2a}{3}$

$$\text{Max Volume} = \pi b\left(\frac{2a}{3}\right)^2 - \frac{\pi b}{a}\left(\frac{2a}{3}\right)^3$$
$$= \frac{4\pi ba^2}{9} - \frac{8\pi b}{a}\frac{a^3}{27}$$
$$= \frac{4\pi ba^2}{9} - \frac{8\pi ba^2}{27}$$
$$= \frac{4\pi ba^2}{27}$$

iv) $V_{cone} = \frac{1}{3}\pi a^2 b$

$$\frac{4}{9}\left(\frac{1}{3}\pi a^2b\right) = \frac{4}{27}\pi a^2 b$$

$\therefore$ Cylinder with max volume is $\frac{4}{9}$ that of cone