Sydney Technical High School

Extension One Mathematics
HSC Assessment Task 2
March 2011

Name...

Teacher..

General Instructions

- Working Time – 70 minutes.
- Write using a blue or black pen.
- Approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a new page.

Total marks (60)

- Attempt Questions 1-6.
- All questions are of equal value.
- Mark values are shown with the questions

Question	1	2	3	4	5	6	TOTAL
Mark							

Question 1 (10 marks) **Marks**

a) Find the primitive function of $\dfrac{3}{4\sqrt{x}}$ **1**

b) Consider the curve $y = x^4 - \dfrac{4}{3}x^3 - 2x^2 + 4x + 3$

(i) Obtain y' and y'' for this function **2**

(ii) Find the stationary points. **2**

(iii) Determine the nature of each of the stationary points. **2**

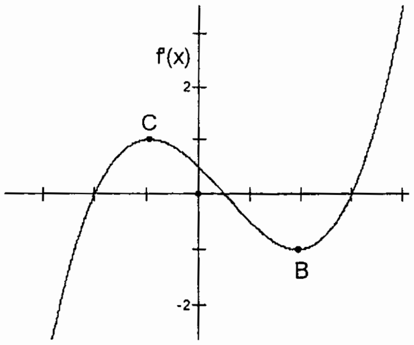
(iv) Find the x coordinates of the two points of inflexion. **1**

(v) Sketch the curve for the domain **2**

Question 2 (10 marks) Begin a SEPARATE sheet of paper

a) The graph of $y = f'(x)$ is shown. The zeros of $f'(x)$ are $x = -2, 0.5$, and 3

C has x coordinate -1 and B has x coordinate 2



(i) For what values of x is $f(x)$ increasing? **1**

(ii) C is a local maximum on $f'(x)$.

What type of point occurs on $f(x)$ at the same x value as that shown at C.

Justify your answer. **2**

(iii) For what values of x is $f(x)$ concave down? **1**

c) $g'(x) = 3x^2 - 4 + \dfrac{1}{x^2}$ **3**

$g(x)$ takes the value 4 when $x = 1$. Find $g(x)$.

d) Evaluate $\displaystyle\int_1^2 \left(x^2 + \frac{1}{x^3}\right)dx$ **3**

Question 3 (10 marks) Begin a SEPARATE sheet of paper **Marks**

a) $y = f(x)$ is a continuous function and has a table of values as shown below. **4**

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	2.3	2.5	3.1	2.7	2.4	2.1	1.6

Use the Trapezoidal rule to find the approximate value of $\int_1^4 f(x)\,dx$ correct to one decimal place.

b) Two sailors are paid to bring a motor launch back to Sydney from Gilligans Island, a distance of 1 200 km. They are each paid \$25 per hour for the time spent at sea.

The launch uses fuel at a rate $R = 20 + \frac{v^2}{10}$ litres per hour. Diesel costs \$1.25 per L and (v) is the velocity in km/hour.

(i) Show that, to bring the launch back from Gilligans Island, the total cost to the owners is $\frac{90000}{v} + 150v$. **3**

(ii) Find the speed which minimises the cost and determine this cost. **3**

Question 4 (10 marks) Begin a SEPARATE sheet of paper

a) Use Simpson's rule with 5 function values to evaluate **3**

$$\int_0^4 \frac{\sqrt{144 - 9x^2}}{4}\,dx$$

b) Consider the functions $y = 3 - \frac{x}{2}$ and $y = \frac{1}{2}x^2 - 2x + 1$

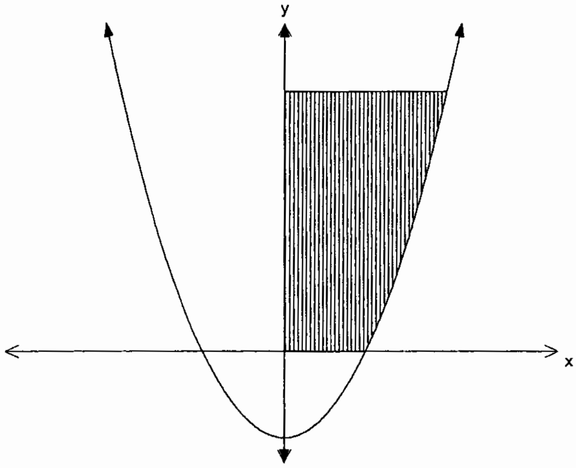
(i) Find the x values where the curves intersect. **2**

(ii) Find the area between the curves. **2**

c) Using the substitution $u = 2x^2 - 3x$, or otherwise, find $\int \frac{(4x-3)dx}{\sqrt{2x^2 - 3x}}$ **3**

Question 5 (10 marks) Begin a SEPARATE sheet of paper **Marks**

a) The diagram shows the region bounded by the curve $y = 2x^2 - 2$, the line $y = 6$ and the x and y axes. **4**

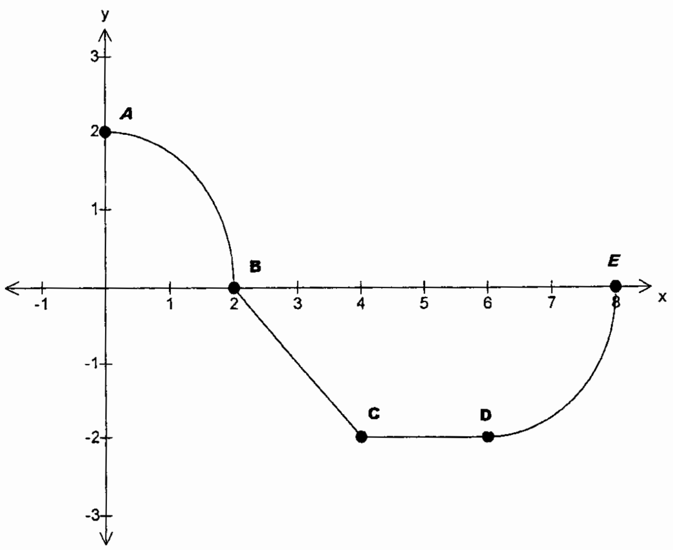


Find the volume of the solid of revolution formed when the region is rotated about the y axis.

b) Evaluate $\int_3^{18} \frac{x}{\sqrt{x-2}}\,dx$ using a suitable substitution. **3**

c) The region, enclosed by the parabola $y^2 = 4ax$ and the line $x = a$, is rotated about the x-axis. Find the volume of the solid formed. **3**

Question 6 (10 marks) Begin a SEPARATE sheet of paper

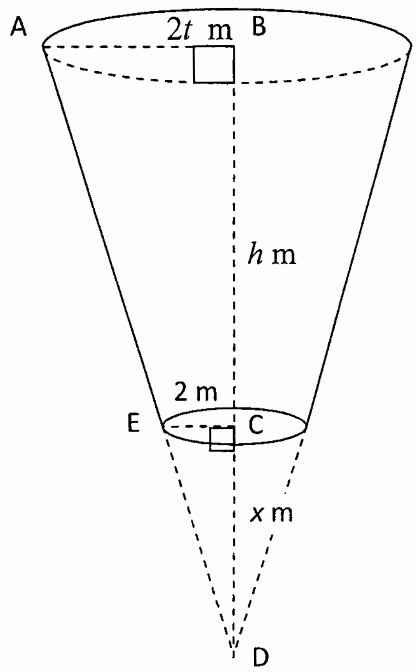


a) The graph of the function f consists of a quarter circle AB, a straight line segment BC, a horizontal straight line segment CD, and a quarter circle DE as shown above.

(i) Evaluate $\int_0^8 f(x)dx$ **2**

(ii) For what values of x satisfying $0 < x < 8$ is the function f NOT differentiable **1**

b) A truncated cone is to be used as a part of a hopper for a grain harvester. It has a total height of h metres. The top radius is to be t times greater than the bottom radius which is 2 metres.



AB = $2t$ metres

BC = h metres

EC = 2 metres

i) If x is the height of the removed section of the original cone, show using similar triangles that $x = \dfrac{h}{t-1}$ **2**

ii) Show that the volume of the truncated cone is given by **2**

$$V = \left(\frac{4\pi h}{3}\right)\left(t^2 + t + 1\right)$$

iii) If the upper radius plus the lower radius plus the height of the truncated cone must total 12 metres, calculate the maximum volume of the hopper. **3**

END OF EXAMINATION

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Question 1

i) $\int \frac{3}{4\sqrt{x}}\,dx = \int \frac{3}{4} x^{-\frac{1}{2}}$

$= \frac{3\sqrt{x}}{2} + c$ (1)

ii) i) $y = x^4 - \frac{4}{3}x^3 - 2x^2 + 4x + 3$

$y' = 4x^3 - 4x^2 - 4x + 4$ (1)

$y'' = 12x^2 - 8x - 4$ (1)

ii) $4x^3 - 4x^2 - 4x + 4 = 0$

$x^2(x-1) - 1(x-1) = 0$

$(x-1)(x+1)(x-1) = 0$

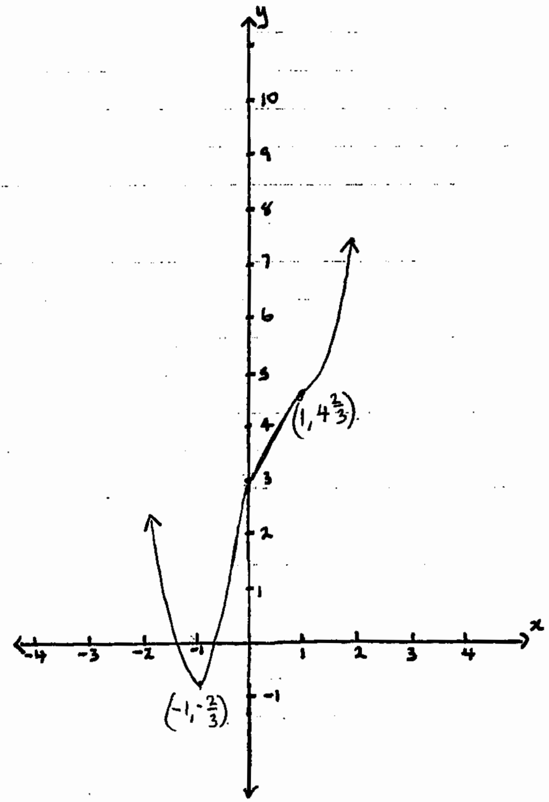
$\left\{\begin{matrix} x = 1 & x = -1 \\ y = 4\frac{2}{3} & y = -\frac{2}{3} \end{matrix}\right\}$ (1) (1)

iii) When $x = -1$ $y'' > 0$

$\therefore$ minimum at $(-1, -\frac{2}{3})$ (1)

When $x = 1$ $y'' = 0$

$\therefore$ horizontal point of inflexion at $(1, 4\frac{2}{3})$ (1)



v) Points of Inflexion occur when $y'' = 0$

$12x^2 - 8x - 4 = 0$

$3x^2 - 2x - 1 = 0$

$(3x+1)(x-1) = 0$

$x = -\frac{1}{3}$ or 1 (1)

Question 2

a i) $f(x)$ is increasing where $f'(x) > 0$
ie $-2 < x < \frac{1}{2}$ and $x > 3$ (1)

ii) A point of inflexion, since c has max. gradient between $x = -2$ and $x = 0.5$ which are stat points $(f'(x) = 0)$ (1)

iii) $f(x)$ will be concave down when $f'(x)$ is decreasing
$-1 < x < 2$ (1)

b. $g(x) = \int g'(x)\,dx$

$= x^3 - 4x - x^{-1} + c$ (1)

$g(1) = 4$

$4 = 1^3 - 4 \times 1 - 1^{-1} + c$

$c = 8$ (1)

$g(x) = x^3 - 4x - \frac{1}{x} + 8$ (1)

c. $\int_1^2 \left(x^2 + \frac{1}{x^3}\right) dx$

$= \left[\frac{x^3}{3} - \frac{1}{2x^2}\right]_1^2$ (1)

$= \left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$ (1)

$= \frac{65}{24}$ (1)

$= 2\frac{17}{24}$

Question 3

a) $\int_1^4 f(x)\,dx \approx \frac{h}{2}\left[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 \ldots\right.$

$\approx \frac{0.5}{2}\left[2.3 + 1.6 + 2(2.5 + 3.1 + 2.7 + 2.4 + 2.1)\right]$

$\approx \frac{1}{4}\left[3.9 + 2(12.8)\right]$ (1)

$\approx \frac{1}{4}\left[3.9 + 25.6\right]$ (1)

≈ 7.4 unit^2 (1 dp) (1)

b i) Time to complete the trip $\frac{1200}{v}$ and sailors paid \$50/h

$\text{Cost} = \left[20 + \frac{v^2}{10}\right] \times \frac{1200}{v} \times 1.25 + 50 \times \frac{1200}{v}$

$\text{Cost} = \frac{1200}{v}\left[75 + \frac{1.25v^2}{10}\right]$

$\text{Cost} = \frac{90000}{v} + 150v$ (1)

b ii) $\frac{d(\text{cost})}{dv} = 150 - \frac{90\,000}{v^2} = 0$

When $v^2 = 600$

$v = 24.495$ km/h (1)

$\frac{d^2(\text{cost})}{dv^2} = 180\,000v^{-3}$ at $v = 24.495$

$\frac{180\,000}{24.495} > 0$

$\therefore$ min (1)

$\therefore \text{Cost} = \frac{90000}{24.495} + 150 \times 24.95$

$= \$7348.47$

Question 4

$\approx \frac{h}{3}\left[f(0)+f(4)+2\times f(2)+4\left(f(1)+f(3)\right)\right]$

$\approx \frac{1}{3}\left[3+0+\frac{2\times\sqrt{108}}{4}+4\times\left(\frac{\sqrt{135}}{4}+\frac{\sqrt{63}}{4}\right)\right]$

≈ 9.2507855

i) $y=3-\frac{x}{2}$

$y=\frac{1}{2}x^2-2x+1$

$3-\frac{x}{2}=\frac{x^2}{2}-2x+1$ (1)

$6-x=x^2-4x+2$

$0=x^2-3x-4$

$0=(x-4)(x+1)$

$x=-1 \quad x=4$ (1)

ii) $\int_{-1}^{4} 3-\frac{x}{2}-\left(\frac{x^2}{2}-2x+1\right)dx$

$=\int_{-1}^{4} 2+\frac{3x}{2}-\frac{x^2}{2}\,dx$

$=\left[2x+\frac{3x^2}{4}-\frac{x^3}{6}\right]_{-1}^{4}$

$=\left(8+12-\frac{32}{3}\right)-\left(\frac{3}{4}+\frac{1}{6}-2\right)$

$=10\frac{5}{12}$

4c) $\int\frac{4x-3}{\sqrt{2x^2-3x}}\,dx$

$u=2x^2-3x$

$\frac{du}{dx}=4x-3$ (1)

$\therefore \int\frac{4x-3}{\sqrt{2x^2-3x}}=\int\frac{du}{\sqrt{u}}=\int u^{-\frac{1}{2}}du$

$=2u^{\frac{1}{2}}+c$

$=2(2x^2-3x)^{\frac{1}{2}}+c$

$=2\sqrt{2x^2-3x}+c$

Question 5

1a) $y=2x^2-2$

$V=\pi\int_0^6 x^2\,dy$ (1)

$=\pi\int_0^6\frac{y+2}{2}\,dy$

$=\pi\left[\frac{y^2}{4}+y\right]_0^6$ (1)

$=\pi\left[\left(\frac{36}{4}+6\right)-0\right]$

$=15\pi$ units^3 (1)

b) $\int_3^{18}\frac{x}{\sqrt{x-2}}\,dx$

let $u=\sqrt{x-2}$

$u^2=x-2$

$2u\frac{du}{dx}=1$

$dx=2u\,du$

$x=u^2+2$

when $x=3 \quad u=1$

$x=18 \quad u=4$

$=2\int_1^4\frac{(u^2+2)\,u\,du}{u}$

$=2\int_1^4 u^2+2\,du$

$=2\left[\frac{u^3}{3}+2u\right]_1^4$

$=54$

Question 5 continued

5c) $V=\pi\int_0^a y^2\,dx$ (1)

$V=\pi\int_0^a 4ax\,dx$

$V=\pi\left[2ax^2\right]_0^a$ (1)

$V=\pi\left[2a^3-0\right]$

$V=2\pi a^3$ units^3 (1)

Question 6

a i) $\int_0^8 f(x)\,dx=-\left(\frac{1}{2}\times2\times2\right)-2\times2$

$=-6$ (2)

a ii) The function is NOT differentiable at $x=2$ and $x=4$

(the end points are NOT included at $x=6$, the gradient is continuous) (1)

b i) In Δ ABD and Δ ECD

$\frac{2t}{h+x}=\frac{2}{x}$

$2tx=2(h+x)$ (1)

$2tx=2h+2x$

$2tx-2x=2h$

$2x(t-1)=2h$

$x=\frac{h}{t-1}$ (1)

b ii) $V=\frac{1}{3}\pi(2t)^2\cdot(h+x)-\frac{1}{3}\pi2^2x$

$=\frac{1}{3}\pi(2t)^2\left(h+\frac{h}{t-1}\right)-\frac{1}{3}\pi(2)^2\left(\frac{h}{t-1}\right)$ (1)

$=\frac{1}{3}\pi(2t)^2\left(\frac{ht}{t-1}\right)-\frac{1}{3}\pi(2)^2\left(\frac{h}{t-1}\right)$

$=\frac{1}{3}\pi(2)^2\left(\frac{h}{t-1}\right)(t^3-1)$

$=\frac{4}{3}\pi\left(\frac{h}{t-1}\right)(t-1)(t^2+t+1)$

$=\frac{4\pi h}{3}(t^2+t+1)$ (1)

ii) Sum of radii and height $=12$

$2+h+2t=12$

$h=10-2t$ (1)

$V=\frac{4\pi h}{3}(t^2+t+1)$

$V=\frac{4\pi}{3}(10-2t)(t^2+t+1)$ (1)

$V=\frac{4\pi}{3}(10t^2+10t+10+10-2t^3-2t^2-2t$

$V=\frac{4\pi}{3}(8t^2+8t-2t^3+10)$ (1)

$\frac{dV}{dt}=\frac{4\pi}{3}(16t+8-6t^2)=0$

$16t+8-6t^2=0$

$t=\frac{-16\pm\sqrt{16^2-4\times-6\times8}}{2\times-6}$

$t=\frac{-16\pm\sqrt{448}}{-12}$

$t=-0.43$ or 3.10

$\frac{d^2V}{dt}=\frac{4\pi}{3}(16-12(3.10)$

$=-88.7$

$\therefore \frac{d^2V}{dx^2}<0$

$\therefore V$ is a maximum

$V=\frac{4\pi}{3}\left[8\times3.10^2+8\times3.10-2\times3.10^3+10\right]$

$=218.2$