Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics Extension 1

HSC Course

Assessment 2

March, 2016

Time allowed: 90 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A BOSTES Reference Sheet is provided at the rear of this Question Booklet, and may be removed.

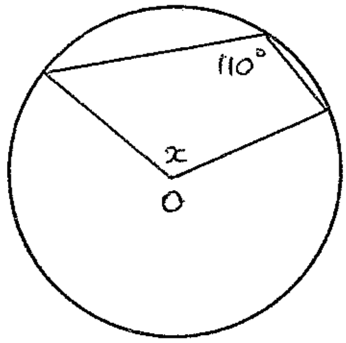
Section 1 Multiple Choice
Questions 1-6
6 Marks

Section II Questions 7-12
60 Marks

QUESTION 1

O is the centre. Diagram is not to scale.

What is the size of x in degrees?

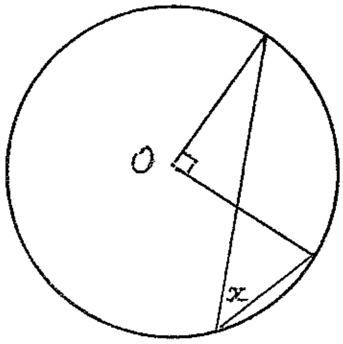


A. 110 B. 120 C. 130 D. 140

QUESTION 2

O is the centre. Diagram is not to scale.

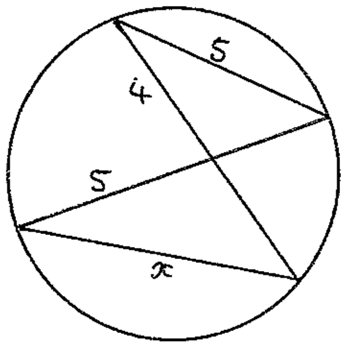
What is the size of x in degrees?



A. 30 B. 45 C. 60 D. 75

QUESTION 3

Diagram is not to scale. What is the length of x in units?



A. 6 B. 6.25 C. 6.5 D. 6.75

QUESTION 4

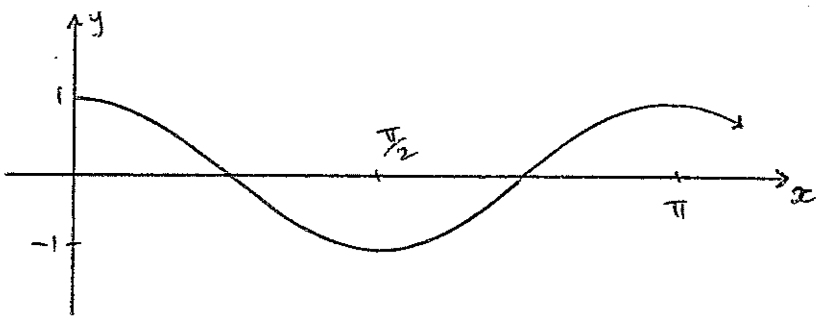
Which of the following is **not** a possible function for this curve?

A. $y = \cos 2x$

B. $y = \sin 2\left(x - \frac{3\pi}{4}\right)$

C. $y = \sin\left(2x + \frac{\pi}{4}\right)$

D. $y = -\cos 2\left(x - \frac{\pi}{2}\right)$



QUESTION 5

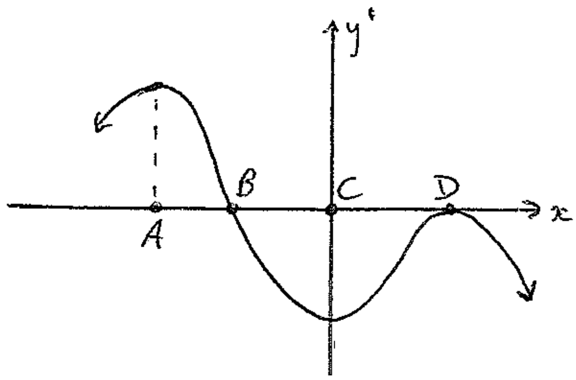
For which values of x is the curve $y = x^3 - 3x^2$ both concave up and decreasing?

A. $x > 1$ B. $1 < x < 2$ C. $0 < x < 2$ D. $0 < x < 1$

QUESTION 6

The graph of a derivative function is shown.

At which x value on the original curve $y = f(x)$ will a horizontal point of inflexion occur?



A. $x = A$

B. $x = B$

C. $x = C$

D. $x = D$

QUESTION 7 (10 marks) **Start a new page.**

a) Write the exact value of $\sec\frac{5\pi}{4}$ 1

b) i) Differentiate $x(2x+1)^3$ 1

ii) Find the equation of the tangent to the curve $y = x(2x+1)^3$ at the point where $x = -1$. 2

c) Differentiate $\tan(\sin 2x)$ 2

d) Solve $3cos^2x = \cos x$ for $0 \le x \le 2\pi$. Give solutions correct to 2 decimal places where necessary. 3

e) Find $\int \frac{x^2+1}{x^2}\,dx$ 1

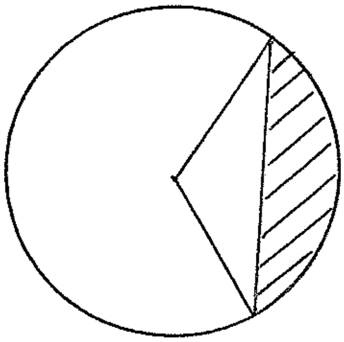
QUESTION 8 (10 marks) **Start a new page.**

a) Find $\int (mx-k)^3\,dx$ 1

b) The curve $y = ax^2 + bx$ has a gradient of 6 at (2,0). Find the values of a and b. 2

c) The area of the shaded segment is 100 cm^2 and subtends an angle of $\frac{5\pi}{6}$ radians at the centre. 2

Find the radius, correct to 1 decimal place.



d) Given the curve $y = 1 - 2\cos\frac{\pi x}{5}$, x in radians.

i) Find the period of this curve. 1

ii) Find x intercepts for one period, $x \geq 0$. 2

iii) Sketch the curve for one period, $x \geq 0$. Show x and y intercepts. 2

QUESTION 9 (10 marks) **Start a new page.**

a) Using a neat diagram and a ruler, find an approximate solution to the equation 2

$\sin\frac{x}{2} = 1 - \frac{x}{3}$

b) For the curve $y = \frac{-3x}{(x-1)^2}$:

i) What is the equation of the vertical asymptote? 1

ii) Find the stationary point and determine its nature. 3

iii) Examine the behaviour of the curve as $x \to \pm\infty$ 1

iv) Neatly sketch the curve. Use a ruler and show relevant features. 2

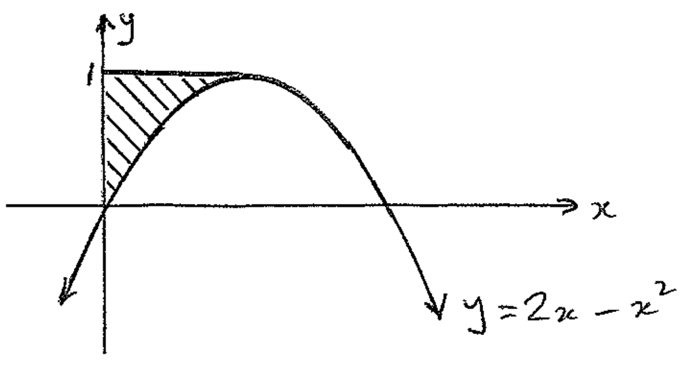
v) Indicate, with an arrow, the location of a point of inflexion on the curve. 1

<u>Do not find its coordinates.</u>

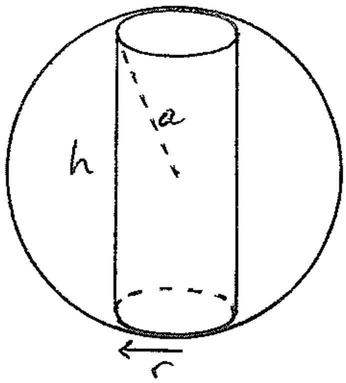
QUESTION 10 (11 marks) **Start a new page.**

a) Evaluate $\int_{-\sqrt{7}}^{0} x\sqrt{16 - x^2}\, dx$, using the substitution $u = 16 - x^2$ 3

b) Find the shaded area: 3



c) Consider the largest cylinder with height h units and radius r units that can fit inside a sphere of fixed radius a units.



i) Show that the cylinder has volume $V = \frac{\pi}{4}(4a^2h - h^3)$. 2

ii) Prove that the maximum volume of the cylinder is $\frac{4\pi a^3}{3\sqrt{3}}\ u^3$ 3

QUESTION 11 (10 marks) **Start a new page.**

a) Use Mathematical Induction to prove that $9^n - 5^n$ is always divisible by 4, for all positive integers n. 3

b) i) On the same axes, sketch the curves $y = x^2$ and $x = y^2$. Show points of intersection. 1

ii) Find the magnitude of the area bounded by the two curves. 3

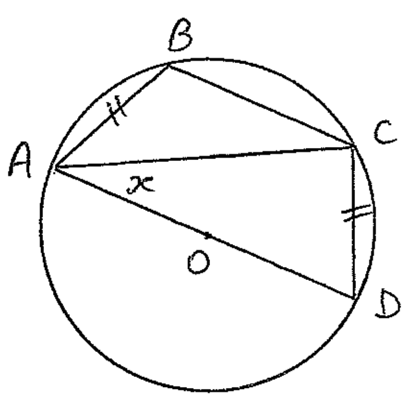
c) The area between $y = 1 - x^2$ and the coordinate axes is rotated about the y axis. 3

Find the volume of the generated solid.

QUESTION 12 (9 marks) **Start a new page.**

a) A, B, C, D are points on the circle, centre O.

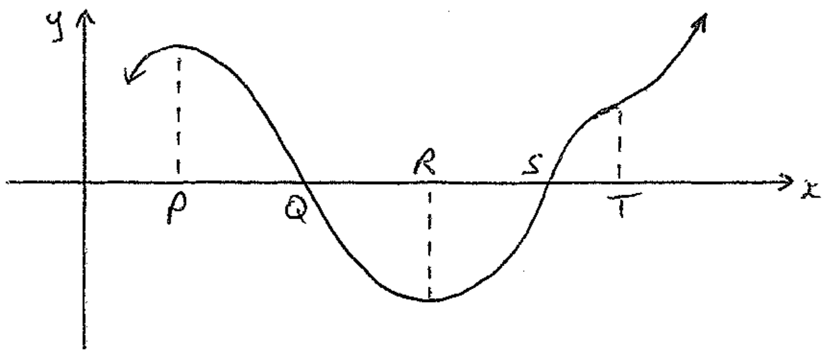
A B = C D



i) Neatly redraw the diagram into your answer booklet.

ii) If $\angle DAC = x$ degrees, find the size of $\angle BAC$ in terms of x. Give reasons. 3

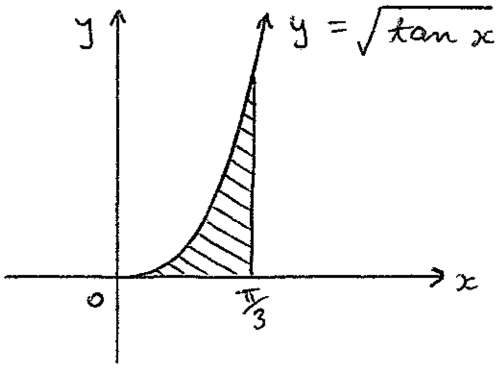
b) The curve shows a function $y = f(x)$.



Neatly sketch a possible curve for $y = f''(x)$.

Show corresponding positions P to T on your graph. 3

c) The shaded area below is rotated about the x axis.



Use Simpson's Rule and five function values to estimate the volume of the generated solid. Give your answer correct to one decimal place. 3

END OF EXAM

SOLUTIONS

① D ② B ③ B ④ C ⑤ B ⑥ D

⑦ a) $\frac{1}{\cos 225°} = \frac{-1}{\cos 45°}$

$= -\sqrt{2}$

b) i) $y' = 1(2x+1)^3 + 3(2x+1)^2 \times 2 \times x$

$= (2x+1)^3 + 6x(2x+1)^2$

ii) $x = -1 \Rightarrow y' = -1 - 6(1)$

$m_T = -7$ and $(-1, 1)$

$\therefore$ tangent is $y - 1 = -7(x+1)$

$y = -7x - 6$

c) $y' = \sec^2(\sin 2x) \times \cos 2x \times 2$

$= 2\sec^2(\sin 2x)\cos 2x$

d) $3\cos^2 x - \cos x = 0$

$\cos x(3\cos x - 1) = 0$

$\cos x = 0$ or $\frac{1}{3}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}, 1.23, 5.05$

e) $\int (1 + x^{-2})\,dx$

$= x - \frac{1}{x} + c$

⑧ a) $\frac{(mx-k)^4}{4m} + c$

b) $y' = 2ax + b$. At $x = 2$, $y' = 6$

$\therefore 6 = 4a + b$ — ①

At $x = 2$, $y = 0$

$\therefore 0 = 4a + 2b$ — ②

② − ① gives $b = -6$

Sub in ② gives $4a - 12 = 0$

$a = 3$

c) $\frac{1}{2}r^2(\theta - \sin\theta) = 100$

$r^2\left(\frac{5\pi}{6} - \frac{1}{2}\right) = 200$

$r^2 = 200 \times \frac{6}{5\pi - 3}$

$r \doteqdot 9.7$ (1 dec.)

d) i) period $= \frac{2\pi}{\pi/5} = 10$

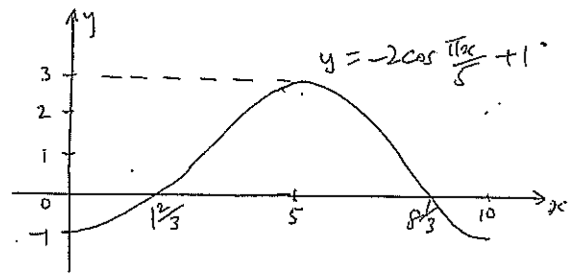
d) ii) $1 - 2\cos\frac{\pi x}{5} = 0$

$\cos\frac{\pi x}{5} = \frac{1}{2}$

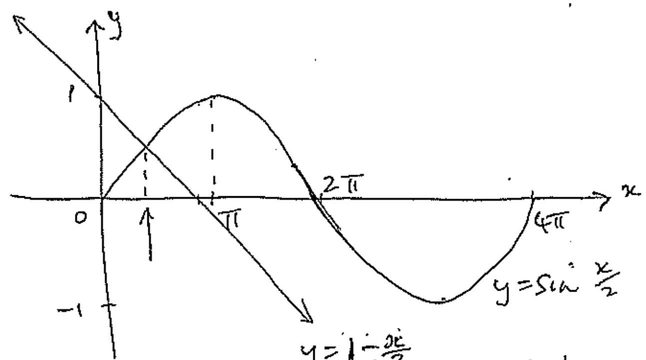
$\frac{\pi x}{5} = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{5}{3}$ or $\frac{25}{3}$

iii)



⑨ a)



Solution is approx

$x = 1.4, 1.5, 1.6$ or $\frac{\pi}{2}$

b) i) $x \neq 1$

ii) S.P. when $y' = \frac{-3(x-1)^2 - 2(x-1)(-3x)}{(x-1)^4} = 0$

$\therefore -3(x^2 - 2x + 1) + 6x^2 - 6x = 0$

$-3x^2 + 6x - 3 + 6x^2 - 6x = 0$

$3x^2 - 3 = 0$

$3(x+1)(x-1) = 0$

$x = \pm 1$ (but $x \neq 1$)

$\therefore x = -1$ only

$y' = \frac{3(x-1)(x+1)}{(x-1)^{4\,3}}$

$= \frac{3(x+1)}{(x-1)^3}$

x	−1.1	−1	−0.9
y'	+	0	−

$\therefore$ max T.P. at $(-1, \frac{3}{4})$

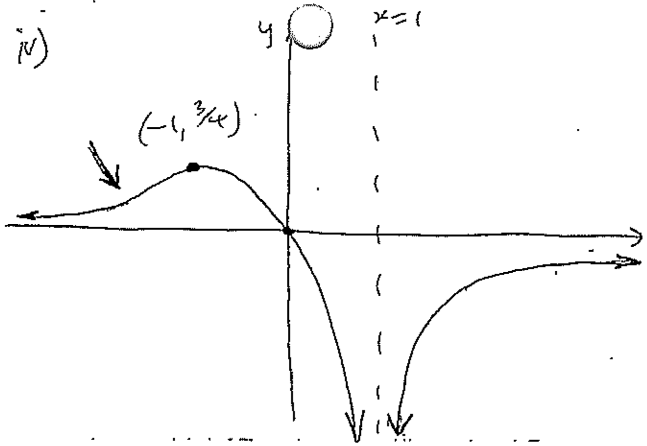
iii) $y = \dfrac{-3x}{x^2-2x+1}$

$= \dfrac{-3/x}{1-\frac{2}{x}+\frac{1}{x^2}}$

As $x \to \pm\infty$,

$y \to 0$

iv)



(10) a) $\int_{-\sqrt{7}}^{0} x\sqrt{16-x^2}\,dx = \int_{9}^{16} x\sqrt{u}\,\dfrac{du}{-2x}$

$u = 16-x^2$, $x=-\sqrt{7}$, $u = 9$

$\dfrac{du}{dx} = -2x$, $x=0$, $u=16$

$dx = \dfrac{du}{-2x}$

$= -\frac{1}{2}\int_{9}^{16} u^{\frac{1}{2}}\,du$

$= -\frac{1}{2}\times\frac{2}{3}\left[u^{3/2}\right]_9^{16}$

$= -\frac{1}{3}(64-27)$

$= -\frac{37}{3}$

b) Area = square − area to x axis

$= 1\times1 - \int_0^1 (2x-x^2)\,dx$

$= 1 - \left[x^2 - \frac{x^3}{3}\right]_0^1$

$= 1 - \left[\left(1-\frac{1}{3}\right)-(0-0)\right]$

$= 1 - \frac{2}{3}$

$= \frac{1}{3}\ u^2$

c) i) $V = \pi r^2 h$

and $a^2 = \left(\frac{h}{2}\right)^2 + r^2$

$\therefore r^2 = a^2 - \frac{h^2}{4}$

$\therefore V = \pi\left(\frac{4a^2-h^2}{4}\right)h$

$= \frac{\pi}{4}\left(4a^2h - h^3\right)\ u^3$

as reqd.

c) ii) max Vol. when $\dfrac{dV}{dh} = 0$

$\dfrac{dV}{dh} = \frac{\pi}{4}\left(4a^2 - 3h^2\right) = 0$

$3h^2 = 4a^2$

$h^2 = \frac{4a^2}{3}$

$h = \frac{2a}{\sqrt{3}}$

Prove max. vol.

$V'' = \frac{\pi}{4}(-6h) < 0$ for all h.

$\therefore$ max. vol. occurs when $h = \frac{2a}{\sqrt{3}}$

and $V_{max} = \frac{\pi}{4}\left(4a^2 \times \frac{2a}{\sqrt{3}} - \frac{8a^3}{3\sqrt{3}}\right)$

$= \frac{\pi}{4}\left(\frac{8a^3}{\sqrt{3}} - \frac{8a^3}{3\sqrt{3}}\right)$

$= \frac{\pi}{4}\left(\frac{24a^3 - 8a^3}{3\sqrt{3}}\right)$

$= \frac{\pi}{4} \times \frac{16a^3}{3\sqrt{3}}$

$= \frac{4\pi a^3}{3\sqrt{3}}\ u^3$ as reqd.

(11) a) Prove true for $n=1$,

$9^1 - 5^1 = 4$, which is divis. by 4.

Assume true for $n=k$, i.e. assume that $9^k - 5^k = 4P$ for some integer P.

Prove true for $n=k+1$, i.e. prove that $9^{k+1} - 5^{k+1} = 4Q$ for some integer Q.

Now, $9^{k+1} - 5^{k+1} = 9\times9^k - 5\times5^k$

$= 9\times(4P + 5^k) - 5\times5^k$ from above

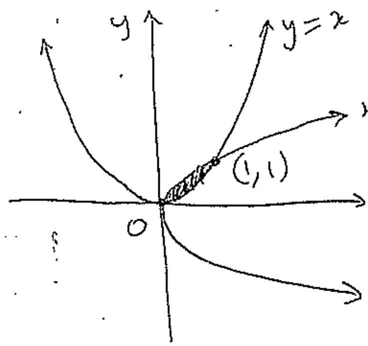
$= 36P + 4\times5^k$

$= 4(9P + 5^k)$

$= 4Q$ since $9P + 5^k$ is integral.

Since the result is true for $n=1$, then it must be true for $n=1+1=2$, then $n=2+1=3$ and so on for all positive integers n.

b) i)



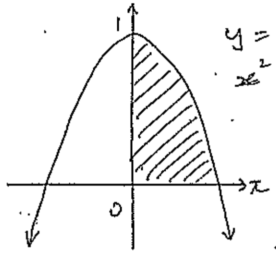
ii) $A = \int_0^1 \left(x^{\frac{1}{2}} - x^2\right) dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$

$= \left(\frac{2}{3} - \frac{1}{3}\right) - (0 - 0)$

$= \frac{1}{3} u^2$

c)



$y = 1 - x^2$

$x^2 = 1 - y$

$V = \pi \int_0^1 (1 - y)\, dy$

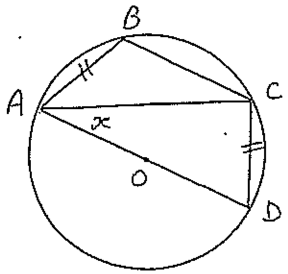
$= \pi \left[y - \frac{y^2}{2}\right]_0^1$

$= \pi \left[\left(1 - \frac{1}{2}\right) - (0 - 0)\right]$

$= \frac{\pi}{2} u^3$.

(12) a)

i)



ii) $\angle ACD = 90°$ (angle in semi circle)

$\angle ACB = x$ (equal angles standing on equal chords)

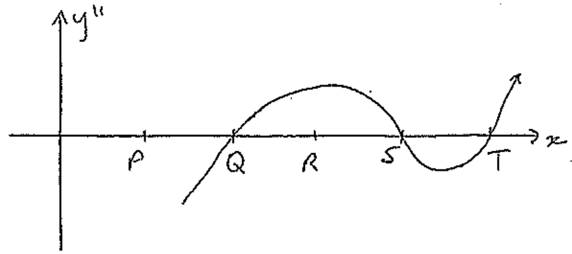
$\therefore \angle BCD = 90 + x$

$\therefore \angle BAC + x = 180 - (90 + x)$ (opposite angles supplementary in cyclic quadrilateral)

$\therefore \angle BAC = 180 - 90 - x - x$

$= 90 - 2x$

b)



c) $V = \pi \int_0^{\frac{\pi}{3}} \tan x\, dx$

$\doteq \pi \times \frac{\pi}{36}\left(\tan 0 + 4\tan\frac{\pi}{12} + 2\tan\frac{2\pi}{12} + 4\tan\frac{3\pi}{12} + \text{ta}\right.$

$= \frac{\pi^2}{36} \times 7.9585$

$= 2.2\ u^3$ (1 dec.)