TARA ANGLICAN SCHOOL FOR GIRLS

2003
YEAR 12
EXTENSION ONE MATHEMATICS

HALF YEARLY EXAMINATION
Weight: 25%

Total Marks: 70 marks

Time Allowed: 1.5 Hours + *5 minutes reading time*

DIRECTIONS TO CANDIDATES

- There are FIVE (5) questions
- Attempt all questions
- Board Approved calculators may be used
- Start a NEW BOOKLET for each question
- All necessary working should be shown in every question. Marks may be deducted for carelessly or badly arranged work.
- An Integral sheet is provided with this paper

(a) If $\sin A = \frac{[illegible]}{\sqrt{7}}$, and $\cos A > 0$, find the exact value of $\sin 2A$. 3

(b) Solve the inequality $\frac{4x+3}{x-4} \geq 1$ 3

(c) Deduce, to the nearest minute, the obtuse angle between the lines $\frac{x}{7}+\frac{y}{5}=1$ and $2x-3y+4=0$. 3

(d) Use the substitution $u = 1 + t$, to find $\int \frac{t}{\sqrt{1+t}}\,dt$ 3

Question Two (13 Marks) START A NEW BOOKLET **Marks**

(a) Integrate the following:

(i) $\int_{-1}^{\frac{1}{2}} \frac{x^2-4x}{x}\,dx$ 2

(ii) $\int (4-y)^3\,dy$ 1.5

(b) (i) Express $7\cos\theta - \sin\theta$, in the form $R\cos(\theta+\alpha)$, where $R > 0$ and $0^\circ \leq \alpha \leq 90^\circ$ 2

(ii) *Hence*, solve $7\cos\theta - \sin\theta = 5$ for $0^\circ \leq \theta \leq 360^\circ$, giving answers to the nearest minute. 2.5

(c) Given $\int_1^K x\sqrt{x} = \frac{62}{5}$, deduce the value of K. 2

(d) Prove $\frac{\sin 2\beta + \sin\beta}{1+\cos 2\beta + \cos\beta} = \tan\beta$ 3

Question Three (19 Marks) **START A NEW BOOKLET** **Marks**

(a) Given $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$,

(i) *Derive* the equation of the tangent to the parabola at P **2**

(ii) The tangent at P and the line through Q parallel to the y – axis, intersect at T. Determine the co-ordinates of T. **2**

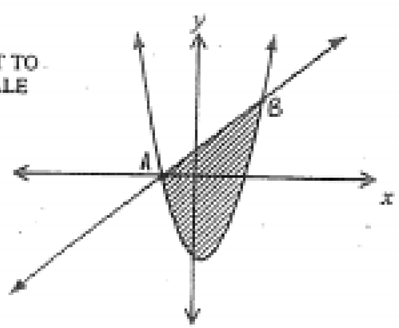
(iii) Calculate the co-ordinates of M, the midpoint of PT. **2**

(iv) Given $pq = -1$, find the equation of the locus of M. **3**

(b) Prove by Mathematical Induction, that $7^n + 11^n$ is divisible by 9, if n is odd. **4**

(c) The diagram below shows the curves $y = x^2 - x - 6$ and $y = x + 2$.

NOT TO SCALE



(i) Find the x – co-ordinates of points A and B. **3**

(ii) Hence, calculate the area enclosed by the two curves. **3**

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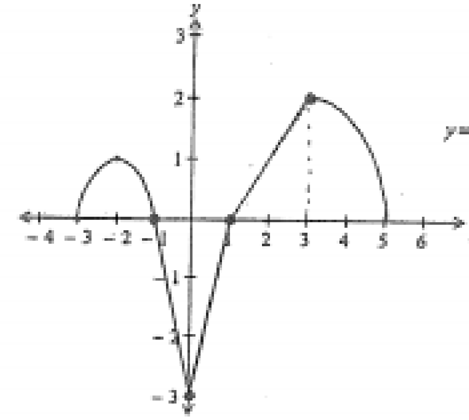


(i) Evaluate $\int_{-3}^{5} h(x)\,dx$ **2**

(ii) Calculate the area enclosed by the curve $y = h(x)$, and the x-axis. **2**

(b) (i) Differentiate $y = \dfrac{2x^2 - 1}{3x^2 + 4}$

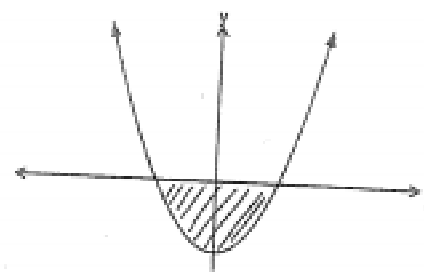
(ii) Hence integrate $\displaystyle\int_0^2 \frac{x\,dx}{(3x^2+4)^2}$ **2**

(c) $P(x)$ is a monic polynomial of degree 4 and has exactly 2 real zeros, at 1 and -1.

(i) If $P(x)$ is an even function find a general equation to represent this information. **2**

(ii) *Hence*, if $P(x) = 33$ when $x = -2$, find the unique polynomial $P(x)$. **2**

(d) The region bounded between $y = x^2 - 1$ and the x-axis is rotated about the x-axis. Determine the exact volume of the solid of revolution formed. **3**



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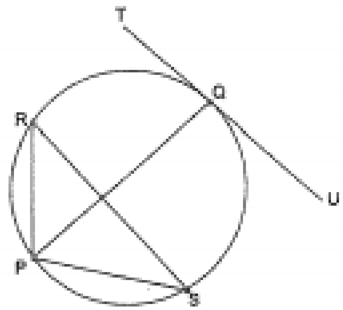
Question Five (11 Marks) **START A NEW BOOKLET** Marks

(a) The diagram below, shows a circle with a chord PQ and another chord RS, which is parallel to the tangent at Q. 4



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Copy or trace the diagram onto your page

Prove that chord PQ bisects $\angle RPS$. [HINT: construction lines may be required]

(b) A vase is formed by the rotation of the curve $y = 4\sqrt{x}$, about the y – axis. Calculate the amount of water needed to fill the vase to a depth of 8 cm. 3

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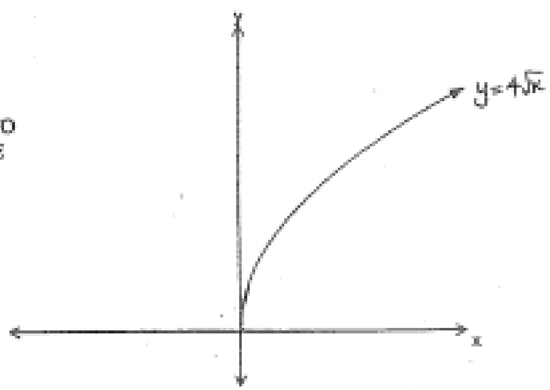

Question Five continued…

(c) The function $f(x)=x^3-x^2-x-1$ is shown below.

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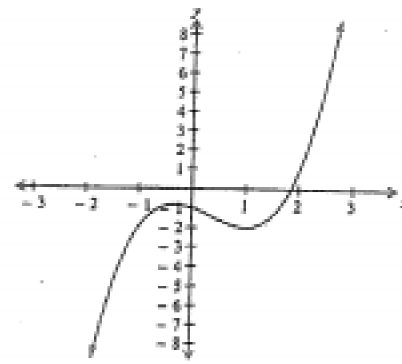


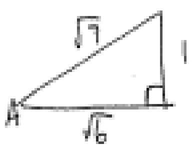
(i) Using x = 2 as a first approximation for $f(x)=0$, use one application of Newton's method to find a better approximation to 1 decimal place. 2

(ii) Copy or trace the diagram. **Describe, in words using your diagram,** why x = 1 is an unsuitable first approximation to this $f(x)=0$. 2

EXTENSION ONE MATHEMATICS - HALF YEARLY 2003 SOLUTIONS

Qu.1

(a) $\sin A = \frac{1}{\sqrt{7}}$ $\cos A > 0$



$\sin 2A = 2\sin A\cos A$

$= 2\left(\frac{1}{\sqrt{7}}\right)\left(\frac{\sqrt{6}}{\sqrt{7}}\right)$

$= \frac{2\sqrt{6}}{7}$

(b) $\frac{4x+3}{x-4} \geq 1$ $x-4 \neq 0$ $\therefore x \neq 4$

$\times$ both sides by $(x-4)^2$

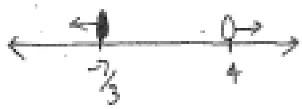
$(x-4)^2 \times \frac{4x+3}{x-4} \geq 1 \times (x-4)^2$

$(x-4)(4x+3) \geq x^2-8x+16$

$4x^2-13x-12 \geq x^2-8x+16$

$3x^2-5x-28 \geq 0$

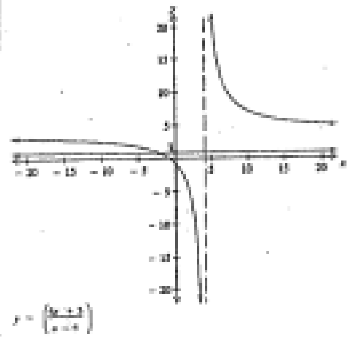
$(3x+7)(x-4) \geq 0$



test $x=0$ $(7)(-4) \geq 0$

$-28 \ngeq 0$

$\therefore x \leq -\frac{7}{3},\ x > 4.$



(c) $\frac{x}{7} + \frac{y}{5} = 1$ $2x-3y+4=0$

$3y = 2x+4$

$m_2 = \frac{2}{3}$

$\frac{y}{5} = 1 - \frac{x}{7}$

$y = 5 - \frac{5x}{7}$

$m_1 = -\frac{5}{7}$

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$ for obtuse angle, do not take absolute value

$\tan\theta = \frac{-\frac{5}{7} - \frac{2}{3}}{1+\left(-\frac{5}{7}\right)\left(\frac{2}{3}\right)}$

$= -\frac{29}{11}$

$\theta = 180° - 69°14'$

$\therefore \theta = 110°46'$

(d) $u = 1+t$

$\frac{du}{dt} = 1$

$\therefore dt = du$

$\int \frac{t}{\sqrt{1+t}}\,dt = \int \frac{u-1}{\sqrt{u}}\,du$

$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$

$= \frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} + C$

$= \frac{2}{3}\sqrt{(1+t)^3} - 2\sqrt{1+t} + C$

Qu 2

(a) (i) $\int_{-1}^{\frac{1}{2}} \frac{x^3-4x}{x}\,dx$

$= \int_{-1}^{\frac{1}{2}} x^2 - 4\,dx$

$= \left[\frac{x^3}{3} - 4x\right]_{-1}^{\frac{1}{2}}$

$= \left(\frac{1}{24} - 4\times\frac{1}{2}\right) - \left(-\frac{1}{3}+4\right)$

$= -5\frac{5}{8}$

(ii) $\int (4-y)^3\,dy = \frac{(4-y)^4}{-4} + C$

(b) (i) $\overset{a}{7}\cos\theta - \overset{b}{\sin}\theta = 5\sqrt{2}\cos(\theta + 8°8')$

$R = \sqrt{7^2+1^2}$

(ii) $5\sqrt{2}\cos(\theta+8°8') = 5$

$\cos(\theta+8°8') = \frac{1}{\sqrt{2}}$

$\theta + 8°8' = 45°, 315°$

$\theta = 36°52', 306°52'$

(c) $\int_1^K x\sqrt{x}\,dx = \frac{62}{5}$

$\int_1^K x^{\frac{3}{2}}\,dx = \frac{62}{5}$

$\left[\frac{2x^{\frac{5}{2}}}{5}\right]_1^K = \frac{62}{5}$

$\frac{2K^{\frac{5}{2}}}{5} - \frac{2}{5} = \frac{62}{5}$

$\frac{2k^{\frac{5}{2}}}{5} = \frac{64}{5}$

$k^{\frac{5}{2}} = 32$

(d) LHS $= \frac{\sin 2\beta + \sin\beta}{1+\cos 2\beta + \cos\beta}$

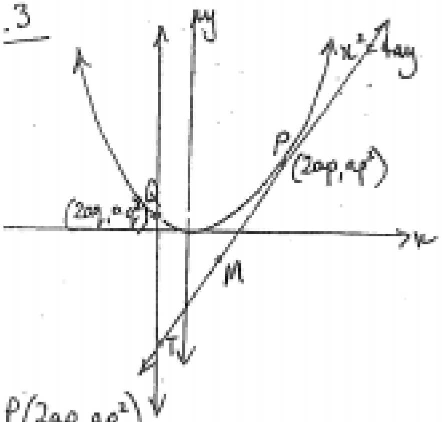
$= \frac{2\sin\beta\cos\beta + \sin\beta}{1+(2\cos^2\beta - 1) + \cos\beta}$

$= \frac{\sin\beta(2\cos\beta+1)}{\cos\beta(2\cos\beta+1)}$

$= \tan\beta$

$=$ RHS.

.3



$P(2ap, ap^2)$

$= 4ay$

$= \frac{x^2}{4a}$

$= \frac{x}{2a}$

$, \frac{dy}{dx} = \frac{2ap}{2a} = p.$

$ap^2 = p(x - 2ap)$

$ap^2 = px - 2ap^2$

$0 = px - y - ap^2$

(ii) $x = 2aq$ ①, $0 = px - y - ap^2$ ②

sub ① into ②

$0 = p(2aq) - y - ap^2$

$0 = 2apq - y - ap^2$

$\therefore y = 2apq - ap^2$

co-ords of T:

$(2aq, 2apq - ap^2)$

(iii) $x = \frac{2ap + 2aq}{2}$

$\therefore x = a(p+q)$

$y = \frac{\cancel{ap^2} + 2apq - \cancel{ap^2}}{2}$

$\therefore y = apq$

$M\left(a(p+q), apq\right)$

(iv) $x = a(p+q)$

$y = apq$ where $pq = -1$ (given

$\therefore y = -a.$

the locus of M is the directrix

(b) $7^n + 11^n$ divisible by 9, n is odd

let $n = 1$,

$7^1 + 11^1 = 18 = 9 \times 2$

$\therefore$ div by 9.

let $n = 3$,

$7^3 + 11^3 = 1674 = 9(186)$

$\therefore$ div. by 9.

Assume true for $n = k$,

$7^k + 11^k = 9M$ where M is an integer

Prove true for $n = k+2$, since n is o

$7^{k+2} + 11^{k+2} = 9P$ where P is integer

$LHS = 7^{k+2} + 11^{k+2}$

cont'd...

$7^2.7^k + 11^2.11^k$

$7^2(9M - 11^k) + 11^2.11^k$

where $7^k + 11^k = 9M$ from assumption

$9 \times 7^2 \times M - 7^2.11^k + 11^2.11^k$

$9 \times 49M + 72.11^k$

$= 9(49M + 8.11^k)$

$= 9P$ where P is an integer,

as required.

ce true for n=1 and
3 and proved true for
k and n=k+2, true
all values of n,

(c) $x^2 - x - 6 = x + 2$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$\therefore x = -2, 4.$

at A, $x = -2$

at B, $x = 4.$

(ii) $A = \int_{-2}^{4} x + 2 - (x^2 - x - 6)\,dx$

$= \int_{-2}^{4} -x^2 + 2x + 8\,dx$

$= \left[-\frac{x^3}{3} + x^2 + 8x\right]_{-2}^{4}$

$= \left(-\frac{64}{3} + 16 + 32\right) - \left(\frac{8}{3} + 4 - 16\right)$

Qu 4

(a) (i) $\int_{-3}^{5} h(x)\,dx = \left(\frac{\pi \times 1^2}{2}\right) - \left(\frac{1}{2} \times 2\right.$

$+ \left(\frac{1}{2} \times 2 \times 2\right)$

$+ \left(\frac{\pi \times}{4}\right.$

$= \frac{\pi}{2} - 3 + 2 + 1$

$= \frac{3\pi}{2} - 1$

(ii) $A = \int_{-3}^{5} h(x)\,dx = \frac{\pi}{2} + 3 + 2$

$= \left(\frac{3\pi}{2} + 5\right)$

4 cont'd...

$$y = \frac{2x^2-1}{3x^2+4}$$

$$\frac{dy}{dx} = \frac{(3x^2+4)(4x) - (2x^2-1)(6x)}{(3x^2+4)^2}$$

$$= \frac{12x^3+16x - 12x^3+6x}{(3x^2+4)^2}$$

$$\frac{dy}{dx} = \frac{22x}{(3x^2+4)^2}$$

$$\int_0^2 \frac{x\,dx}{(3x^2+4)^2} = \frac{1}{22}\left[\frac{2x^2-1}{3x^2+4}\right]_0^2$$

$$= \frac{1}{22}\left[\frac{7}{16} - \left(\frac{-1}{4}\right)\right]$$

$$= \frac{1}{\cancel{22}_{\,2}} \cdot \frac{\cancel{11}}{16}$$

$$= \frac{1}{32}$$

(c)

(i) $P(x) = (x-1)(x+1)(x^2+a^2)$

$\therefore P(x) = (x-1)(x+1)(x^2+a^2)$

(ii) $P(-2) = 33$ where $P(x) = (x^2-1)(x^2+a^2)$

$$33 = (-3)(-1)(4+a^2)$$

$$33 = 3(4+a^2)$$

$$11 = 4+a^2$$

$$7 = a^2$$

$\therefore P(x) = (x^2-1)(x^2+7)$

(d)

$$V = \pi\int_{-1}^{1} y^2\,dx$$

$$= \pi\int_{-1}^{1} (x^2-1)^2\,dx$$

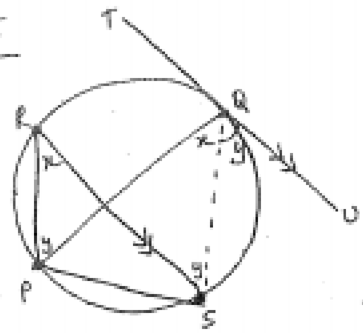
$$= \pi\int_{-1}^{1} x^4-2x^2+1\,dx$$

$$= \pi\left[\frac{x^5}{5} - \frac{2x^3}{3} + x\right]_{-1}^{1}$$

$$= \pi\left[\left(\frac{1}{5} - \frac{2}{3} + 1\right) - \left(\frac{-1}{5} + \frac{2}{3} - \right.\right.$$

$$= \frac{16\pi}{15}\ u^3$$

Qu 5



(a) construction: join QS.

$\angle PRS = \angle PQS$ ($\angle$'s in same segment of circle $=$)
$= x$

$\angle RPQ = \angle RSQ$ ($\angle$'s in same segment of circle $=$)
$= y$

$\angle UQS = \angle QSR$ (alternate $\angle$'s on // lines $=$)
$= y$

$\therefore \angle UQS = \angle QPS$ (alternate segment theorem)
$= y$

$\therefore \angle RPQ = \angle QPS = y$

(b)

$$y = 4\sqrt{x}$$

$$\frac{y}{4} = \sqrt{x}$$

$$\frac{y^2}{16} = x$$

$$V = \pi\int_0^8 x^2\,dy$$

$$= \pi\int_0^8 \frac{y^4}{256}\,dy$$

$$= \pi\left[\frac{y^5}{1280}\right]_0^8$$

$$= \pi\left[\frac{8^5}{1280} - 0\right]$$

$$= \frac{32\,768}{1280}\pi$$

$$\therefore V = \frac{128\pi}{5}\ cm^3$$