TRINITY GRAMMAR SCHOOL

MATHEMATICS DEPARTMENT

YEAR 12 2008 HALF YEARLY EXAMINATIONS

MATHEMATICS EXTENSION 1

(3 UNIT COMPONENT)

ASSESSMENT TASK 3

WEIGHTING 30%

Examination Date:

Friday 2nd May 2008

OUTCOMES REFERRED TO: HE1, HE2, HE3, HE4, HE6, HE7, PE1, PE2, PE3, PE4, PE6.

General Instructions

- Reading time – **5 minutes.**
- Working time – **2 hours.**
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question.
- **Begin** each question in a **new booklet.**
- Write your **examination number** and your **teacher's name** on the front of each answer booklet.

Total marks – 84

- Attempt Questions 1-7.
- All questions are of equal value.
- Mark values are shown at the side of each question part.

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QUESTION 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve the inequation $\dfrac{4}{5-x} \leq 1$. **3**

(b) Use the substitution $u = 1 + \tan x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$. **3**

(c) Find the following:

i) $\displaystyle\int \frac{3x}{x^2+9}\,dx$. **2**

ii) $\displaystyle\int \frac{3}{x^2+9}\,dx$. **2**

iii) $\displaystyle\int \frac{x^2+9}{3x}\,dx$. **2**

QUESTION 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $\dfrac{x+1}{x^2+1} > 1$. **2**

(b) Find $\displaystyle\int x\left(1-x^2\right)^5 dx$, using the substitution $u = 1 - x^2$, or otherwise. **2**

(c) Find the acute angle, correct to the nearest minute, between the lines $3x + y = 4$ and $x - y = 1$. **2**

(d) The point $P(19,-15)$ divides an interval AB externally in the ratio $3:2$. Find the coordinates of the point $B(x,y)$ given $A(-2,3)$. **3**

(e) Prove by mathematical induction that if n is a positive integer, then: **3**

$$\frac{1}{1\times 5}+\frac{1}{5\times 9}+\frac{1}{9\times 13}+\ldots\ldots+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}.$$

QUESTION 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) i) If $f(x) = e^{x+2}$, find the inverse function $f^{-1}(x)$. **2**

ii) On the same axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$. **2**

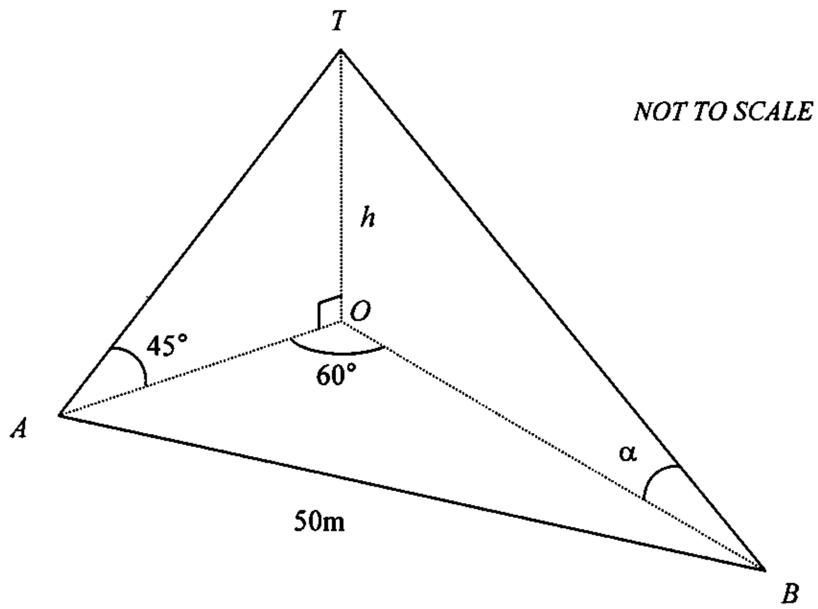
(b) Differentiate $x^2 \cos^{-1}(3x)$. **2**

(c) Find the exact value of $\cos^{-1}\left(\sin\dfrac{4\pi}{3}\right)$. **3**

(d) Determine the domain and range of $y = 2\sin^{-1}\left(\dfrac{x}{3}\right)$ and hence sketch the graph. **3**

QUESTION 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



In the diagram, the points A, B and O are in the same horizontal plane.

A and B are 50m apart and $\angle AOB = 60°$. OT is a vertical tower of height h metres.

The angles of elevation of T from A and B respectively are $45°$ and α.

(α is acute.)

i) Show $AO = h$. **1**

ii) Show $OB = h\cot\alpha$. **1**

iii) By using the cosine rule in triangle AOB, show that: **1**

$h^2\cot^2\alpha - h^2\cot\alpha + h^2 = 50^2$.

iv) Given the tower is 30 high, find the angle α correct to the nearest degree. **2**

(b) Write down the general solution, in terms of π, of the equation $\tan\theta = -\dfrac{1}{\sqrt{3}}$. **2**

(c) α and β are acute angles such that $\cos\alpha = \dfrac{3}{5}$ and $\sin\beta = \dfrac{1}{\sqrt{5}}$. **2**

Without finding the size of either angle, show that $\alpha = 2\beta$.

(d) Without using a calculator, find the exact value of $\sin\left(\cos^{-1}\dfrac{2}{3} + \sin^{-1}\dfrac{1}{4}\right)$. **3**

QUESTION 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Newton's Law of Cooling states that when an object at temperature T (°C) is placed in an environment at a temperature R (°C), then the rate of temperature loss is given by the equation $\frac{dT}{dt} = k(T - R)$; where t is the time in seconds and k is a constant.

A packet of peas, initially at 24°C is placed in a snap-freeze refrigerator in which the internal temperature is maintained at – 40°C. After 5 seconds the temperature of the packet is 19°C. Suppose $T = R + Ae^{kt}$, where A is a constant.

i) State the value of A. **2**

ii) Show that $k = \frac{1}{5}\log_e\left(\frac{59}{64}\right)$. **2**

iii) Hence show that it will take approximately 29 seconds for the packet's temperature to reduce to 0°C. **2**

Question 5 continues on page 7.

QUESTION 5 (continued)

(b) If $y = \frac{1}{200} t e^{-t}$, show that $\frac{dy}{dt} = \frac{1}{200}(1-t)\, e^{-t}$. **1**

(c) Kiran has recently consumed three standard alcoholic drinks.
Immediately after he has finished his last drink, his blood alcohol level is measured over a four-hour period.

Let his blood alcohol level at any time t be A, where t is the time in hours after his last drink.

It is found that the rate of change $\frac{dA}{dt}$ of his blood alcohol content is given by:

$\frac{dA}{dt} = \frac{1}{200}(1-t)\, e^{-t}$, where $0 \le t \le 4$.

i) Show that his blood alcohol content increases during the first hour and decreases after the first hour. **2**

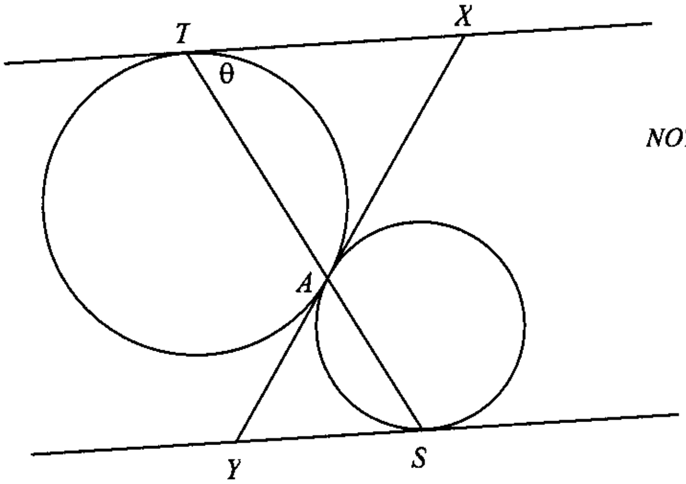
ii) Initially his blood alcohol content was $0 \cdot 0005$. Find A as a function of t. You will need to use part (b). **2**

iii) Determine his maximum alcohol content during the four-hour period. Give your answer correct to four decimal places. **1**

End of Question 5.

QUESTION 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



NOT TO SCALE

In the diagram above, two circles touch one another externally at the point A.

A straight line through A meets one of the circles at T and the other at S.

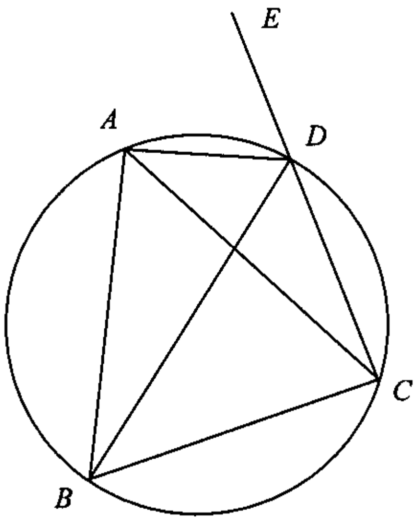
The tangents at T and S meet the common tangent at A at X and Y respectively.

Let $\theta = \angle XTA$.

i) Explain why $\angle XAT$ is θ. **1**

ii) Prove that $TX \parallel YS$. **2**

(b)



NOT TO SCALE

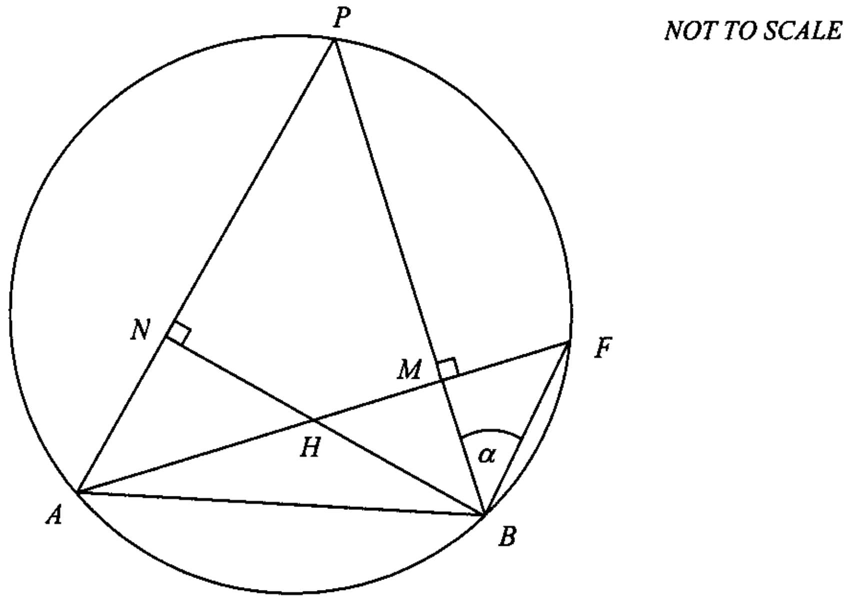
$ABCD$ is a cyclic quadrilateral in which $AB = AC$, and CD is produced to E.

Prove that AD bisects $\angle BDE$. (Hint: let $\angle ADE = \theta$). **3**

Question 6 continues on page 9.

QUESTION 6 (continued)

(c)



In the diagram above, ABP is a triangle inscribed in a circle.

The altitudes BN and AM of the triangle intersect at H.

The altitude AM is produced to meet the circumference of the circle at F.

Copy the diagram into your examination booklet.

Let $\angle PBF = \alpha$.

i) Why is $\angle PAF = \alpha$? **1**

ii) Why are points A, N, M and B concyclic? **1**

iii) Why is $\angle NBM = \alpha$? **1**

iv) Show that M bisects HF. **3**

End of Question 6.

QUESTION 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay \quad (a > 0)$.

i) By derivation, show that the equation of the chord is: **2**

$y = \frac{1}{2}(p+q)x - apq.$

ii) If the chord PQ passes through the focus, S, show that $pq = -1$. **1**

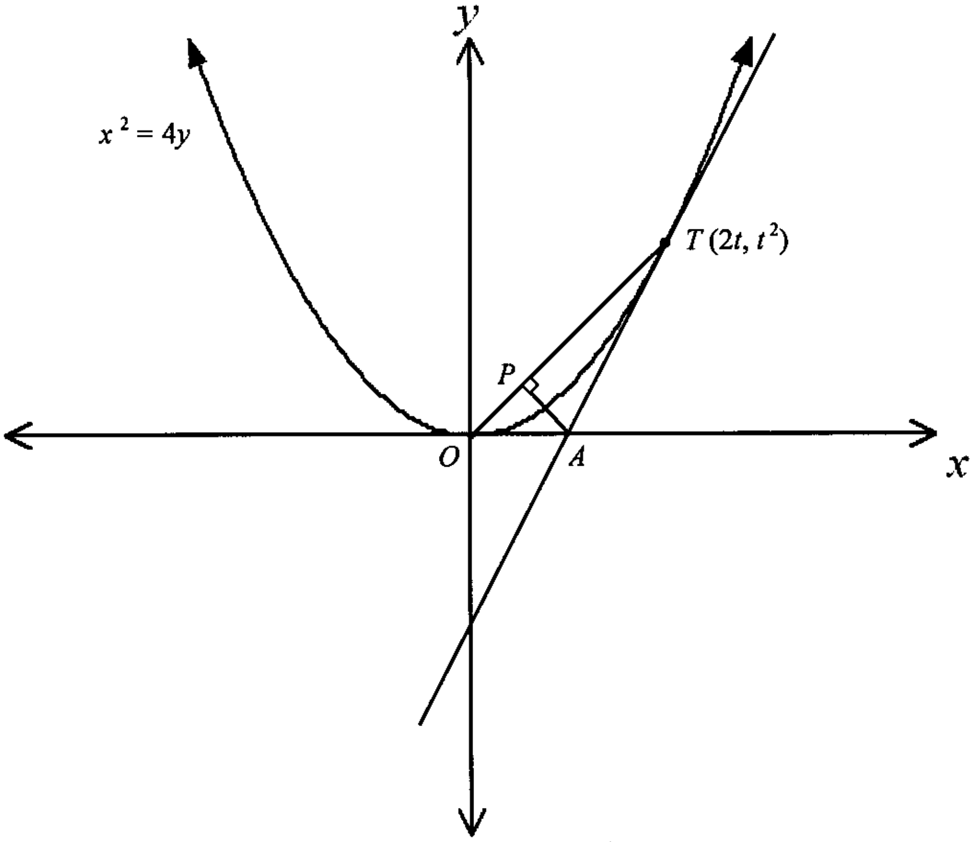
iii) Using the fact that $PQ = PS + SQ$, or otherwise, show that **3**

the chord PQ has length $a\left(p + \frac{1}{p}\right)^2$.

Question 7 continues on page 11.

QUESTION 7 (continued)

(b)



The tangent at $T\left(2t, t^2\right)$, $t \neq 0$, on the parabola $x^2 = 4y$ meets the x-axis at A.

$P(x, y)$ is the foot of the perpendicular from A to OT, where O is the origin.

The equation of the tangent at T is $y = tx - t^2$.

i) Prove that the equation of AP is $y = -\frac{2}{t}(x - t)$. **2**

ii) Show that the equation of OT is $t = \frac{2y}{x}$. **1**

iii) Hence, or otherwise, prove that the locus of $P(x, y)$ lies on a circle with centre $(0,1)$ and give its radius. **3**

End of paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

SOLUTIONS

* MATHEMATICS EXTENSION 1 (3 Unit component)
YEAR 12 - 2008 - HALF YEARLY EXAMINATION

Question 1

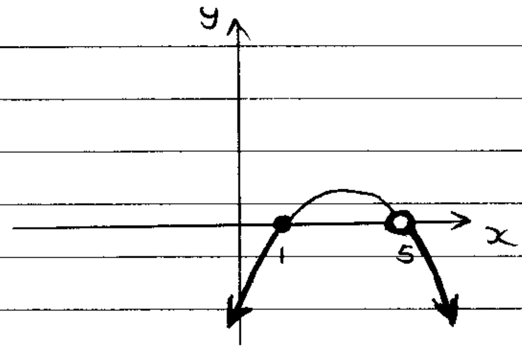
a) $\dfrac{4}{5-x} \leqslant 1 \rightarrow x \neq 5$ (can't ÷ by 0)

$(5-x)^{\cancel{2}} \dfrac{4}{\cancel{5-x}} \leqslant (5-x)^2$ ①

$(5-x)\left[4-(5-x)\right] \leqslant 0$

$(5-x)(x-1) \leqslant 0$ ①

$\therefore x \leqslant 1$ or $x > 5$ ①



b) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$

Let $u = 1 + \tan x$

$\therefore \dfrac{du}{dx} = \sec^2 x$

when $x = 0 \rightarrow u = 1$

$x = \frac{\pi}{4} \rightarrow u = 2$

$= \displaystyle\int_1^2 \frac{1}{\sqrt{u}}\,du$ ①

$= \left[2u^{\frac{1}{2}}\right]_1^2$ ①

$= 2\sqrt{2} - 2$ ①

c) i) $\displaystyle\int \frac{3x}{x^2+9}\,dx = \frac{3}{2}\int \frac{2x}{x^2+9}\,dx$ ①

$= \frac{3}{2}\ln(x^2+9) + C$ ①

c) ii) $\int \frac{3}{x^2+9}\,dx = 3\int \frac{1}{9+x^2}\,dx$

$= 3 \times \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$ ①

$= \tan^{-1}\left(\frac{x}{3}\right) + C$ ①

[full marks for straight to answer in one step also]

iii) $\int \frac{x^2+9}{3x}\,dx = \int \frac{x^2}{3x} + \frac{9}{3x}\,dx$

$= \int \frac{x}{3} + \frac{3}{x}\,dx \quad (x \neq 0)$ ①

$= \frac{x^2}{6} + 3\ln x + C$ ①

Question 2

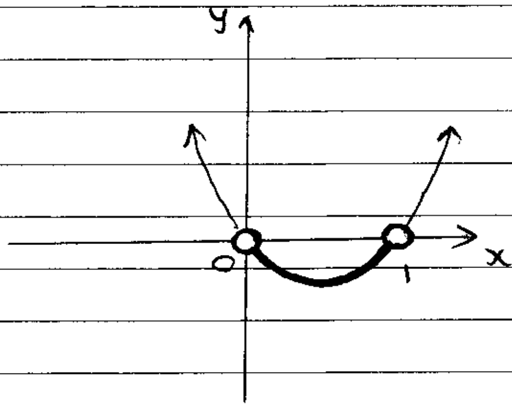
a) $\frac{x+1}{x^2+1} > 1$

$x+1 > x^2+1$ ① (as $x^2+1>0$)

$x^2 - x < 0$

$x(x-1) < 0$

$\therefore\ 0 < x < 1$ ①



b) $\int x(1-x^2)^5\,dx$

$= -\frac{1}{2}\int -2x(1-x^2)^5\,dx$ ①

$= -\frac{1}{2}\int u^5\,du$

$= -\frac{u^6}{12} + C$

$= -\frac{(1-x^2)^6}{12} + C$ ①

Let $u = 1-x^2$

$\therefore \frac{du}{dx} = -2x$

c) $3x + y = 4 \longrightarrow m_1 = -3$

$x - y = 1 \longrightarrow m_2 = 1$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{-3 - 1}{1 + (-3)(+1)} \quad (1)$$

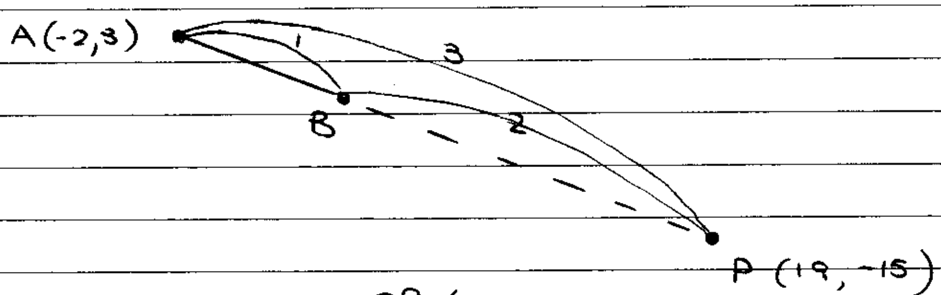
$$= \frac{-4}{-2}$$

$$= 2$$

$\therefore \theta = \tan^{-1}(2)$

$= 63^\circ 26'$ (to nearest min.) (1)

d)



OR

METHOD 1

x-direction:

$x_B = -2 + \frac{1}{3}(19 - (-2))$ (1)

$= 5$

y-direction:

$y_B = 3 - \frac{1}{3}(3 - (-15))$ (1)

$= -3$

$\therefore B(5, -3)$ (1)

METHOD 2

$$x = \frac{mx_2 + nx_1}{m+n}$$

$$19 = \frac{-3(x) + 2(-2)}{-3+2} \quad (1)$$

$-19 = -3x - 4$

$\therefore x = 5$

$$y = \frac{my_2 + ny_1}{m+n}$$

$$-15 = \frac{-3y + 2(3)}{-3+2} \quad (1)$$

$15 = -3y + 6$

$\therefore y = -3$

$\therefore B(5, -3)$ (1)

c) Step 1: Check first case (ie. $n=1$)

$$LHS = \frac{1}{1\times 5}$$

$$= \frac{1}{5}$$

$$RHS = \frac{1}{4\times 1+1}$$

$$= \frac{1}{5}$$

$$= LHS \quad \therefore \text{ true for } n=1$$

Step 2: Assume true for $n=k$

$$\therefore \frac{1}{1\times 5} + \frac{1}{5\times 9} + \frac{1}{9\times 13} + \dots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$$

Step 3: Prove true for $n=k+1$

ie. required to prove $\frac{k}{4k+1} + \frac{1}{(4(k+1)-3)(4(k+1)+1)} = \frac{k+1}{4(k+1)+1}$

$$LHS = \frac{k}{4k+1} + \frac{1}{(4(k+1)-3)(4(k+1)+1)} \quad 1$$

$$= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$$

$$= \frac{k(4k+5)+1}{(4k+1)(4k+5)}$$

$$= \frac{4k^2+5k+1}{(4k+1)(4k+5)}$$

$$= \frac{\cancel{(4k+1)}(k+1)}{\cancel{(4k+1)}(4k+5)} \quad (4k+1 \neq 0)$$

$$= \frac{k+1}{4(k+1)+1} \quad \text{as req'd.} \quad 1$$

Step 4: If true for $n=k$, then true for $n=k+1$.
Since true for $n=1$, then true for $n=2$, $n=3$, $n=4$ and so on...
$\therefore$ True for all positive integer values of n.

1 all stages of induction completed correctly.

Question 3

a) i) $f(x) = e^{x+2}$

$x = e^{y+2}$ (1)

$\ln x = y + 2$

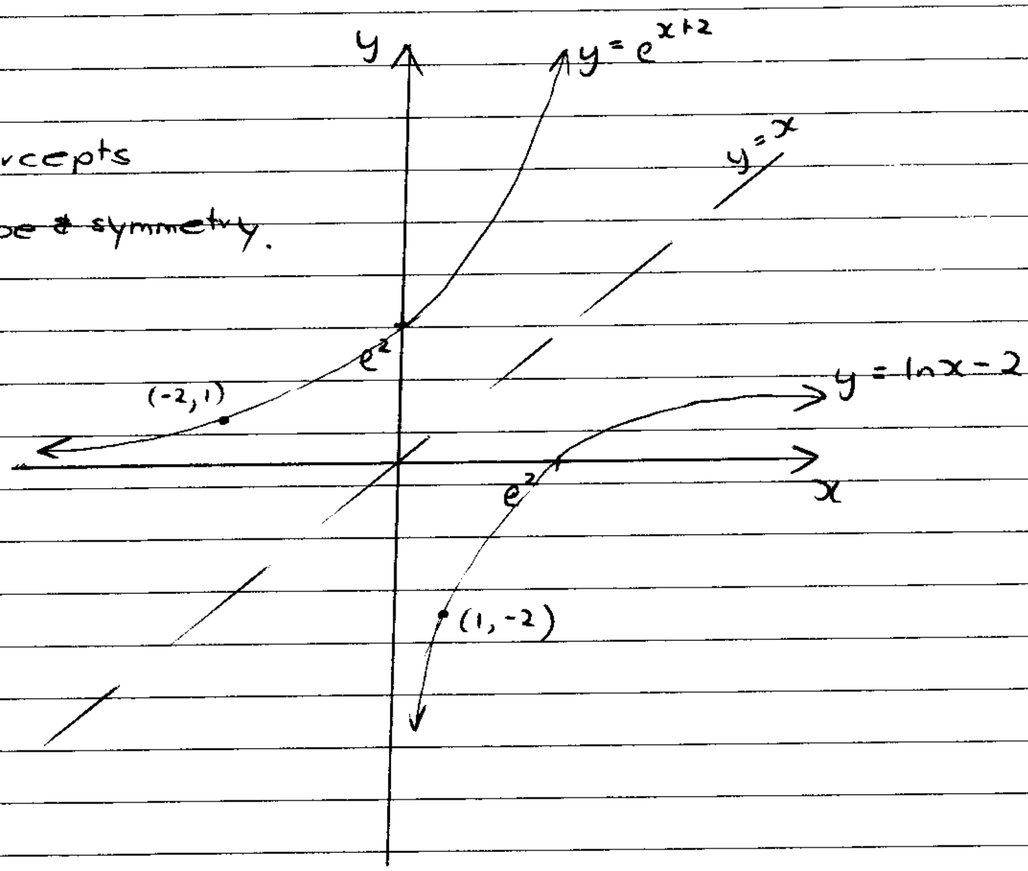
$y = \ln x - 2$

$\therefore f^{-1}(x) = \ln x - 2$ (1)

ii)

(1) intercepts

(1) ~~shape & symmetry.~~



b) $\frac{d}{dx}\left(x^2 \cos^{-1}(3x)\right) = x^2 \cdot \frac{-1}{\sqrt{1-9x^2}} \cdot 3 + \cos^{-1}(3x) \cdot 2x$

$= 2x\cos^{-1}(3x) - \frac{3x^2}{\sqrt{1-9x^2}}$

(1) (1)

c) Let $\alpha = \cos^{-1}\left(\sin\frac{4\pi}{3}\right)$
(where $0 \leq \alpha \leq \pi$)

$\therefore \cos\alpha = -\frac{\sqrt{3}}{2}$ (1)

$\circledast$S | A
$*$T | C

$\therefore \alpha$ in 2nd quadrant (1)

Related $\angle = \frac{\pi}{6}$

$\therefore \alpha = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$ (1)

d) $y = 2\sin^{-1}\left(\frac{x}{3}\right)$

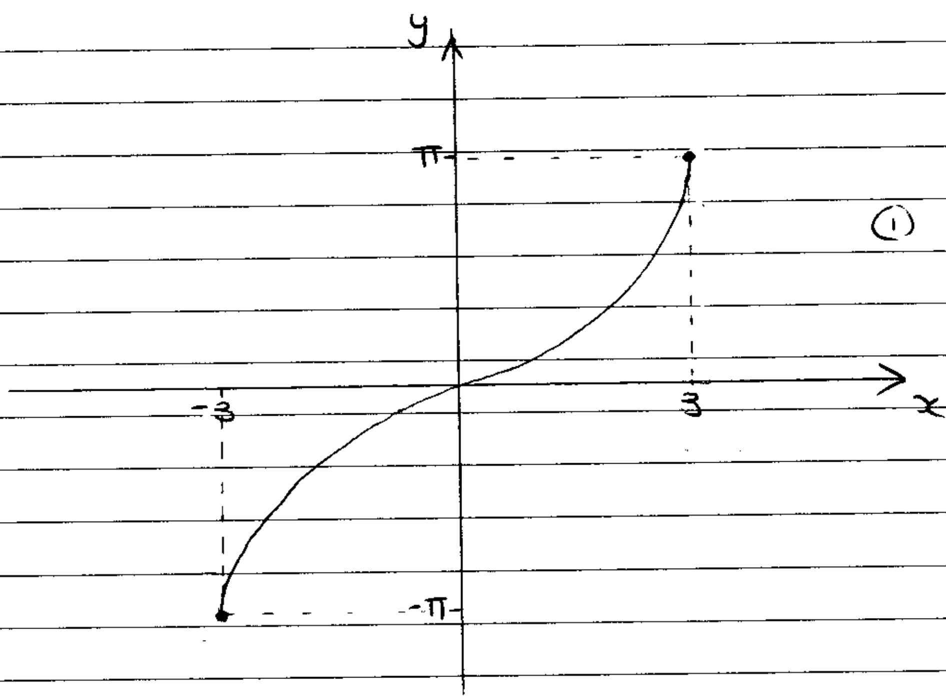
$*$ domain: $-1 \leq \frac{x}{3} \leq 1$

$\therefore -3 \leq x \leq 3$ (1)

$*$ range: $-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{x}{3}\right) \leq \frac{\pi}{2}$

$-\pi \leq 2\sin^{-1}\left(\frac{x}{3}\right) \leq \pi$

$\therefore -\pi \leq y \leq \pi$ (1)



(1)

Question 4

a) i) $\tan 45° = \frac{h}{AO}$

$\therefore AO = \frac{h}{\tan 45°}$

$= h$ (1) (~~or through~~ ISOS. Δ)

ii) $\tan\alpha = \frac{h}{OB}$

$\therefore OB = \frac{h}{\tan\alpha}$

$= h\cot\alpha$ (1)

iii) $AB^2 = AO^2 + OB^2 - 2.AO.OB.\cos 60°$

$50^2 = h^2 + h^2\cot^2\alpha - 2.h.h\cot\alpha.\frac{1}{2}$ (1)

$= h^2 + h^2\cot^2\alpha - h^2\cot\alpha$

Rearranging:

$h^2\cot^2\alpha - h^2\cot\alpha + h^2 = 50^2$ as req'd.

iv) $30^2\cot^2\alpha - 30^2\cot\alpha + 30^2 = 50^2$

$900\cot^2\alpha - 900\cot\alpha + 900 = 2500$

$9\cot^2\alpha - 9\cot\alpha + 16 = 0$ (1)

$\therefore \cot\alpha = \frac{9 \pm \sqrt{81 - 4 \times 9 \times -16}}{18}$

$= \frac{9 \pm \sqrt{657}}{18}$

$\therefore \alpha = 27°$ (to nearest degree) (1)

(as $\alpha > 0$)

b) $\tan\theta = -\frac{1}{\sqrt{3}}$

* S	A
T	C *

related $\angle = \frac{\pi}{6}$

$\therefore \theta = n\pi - \frac{\pi}{6}$ where n is an integer

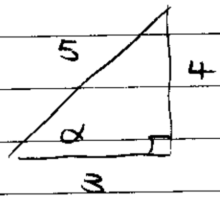
or

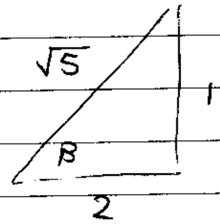
$\theta = n\pi + \frac{5\pi}{6}$ where n is an integer

(1) expression style ie. $n\pi + \ldots$ or $n\pi - \ldots$

(1) correct angle

c) $\cos\alpha = \frac{3}{5}$ $\qquad\qquad$ $\sin\beta = \frac{1}{\sqrt{5}}$





$\sin\alpha = \frac{4}{5}$ 1

$$\sin 2\beta = 2\sin\beta\cos\beta$$
$$= 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}}$$
$$= \frac{4}{5} \quad 1$$
$$= \sin\alpha$$

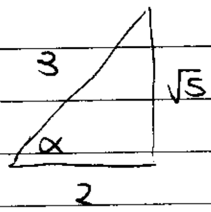
$$\therefore \alpha = 2\beta$$

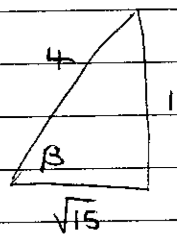
d) $\sin\left(\cos^{-1}\frac{2}{3} + \sin^{-1}\frac{1}{4}\right)$

Let $\alpha = \cos^{-1}\frac{2}{3}$ $\qquad$ & $\qquad$ Let $\beta = \sin^{-1}\frac{1}{4}$

$(0 < \alpha < \pi)$ $\qquad\qquad$ $\left(-\frac{\pi}{2} < \beta < \frac{\pi}{2}\right)$

$\therefore \cos\alpha = \frac{2}{3}$ $\quad$ S | A* / T | C $\therefore$ 1st quad. $\qquad$ $\therefore \sin\beta = \frac{1}{4}$ $\quad$ S | A* / T | C $\therefore$ 1st quad 1





$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad 1$$
$$= \frac{\sqrt{5}}{3} \cdot \frac{\sqrt{15}}{4} + \frac{2}{3} \cdot \frac{1}{4}$$
$$= \frac{\sqrt{75} + 2}{12} \quad 1$$
$$= \frac{5\sqrt{3} + 2}{12}$$

Question 5

a) i) when $t = 0$, $T = 24°C$, $R = -40°C$ 1

$\therefore 24 = -40 + Ae^{0}$ 1

$\therefore A = 64$

ii) $t = 5$, $T = 19°C$

$\therefore 19 = -40 + 64e^{5k}$ 1

$\therefore e^{5k} = \frac{59}{64}$

$\ln \frac{59}{64} = 5k$ 1

$\therefore k = \frac{1}{5} \ln \frac{59}{64}$

iii) $t = ?$ when $T = 0°C$:

$0 = -40 + 64e^{kt}$ 1

$e^{kt} = \frac{40}{64}$

$\ln\left(\frac{5}{8}\right) = kt$

$\therefore t = \dfrac{\ln\left(\frac{5}{8}\right)}{\frac{1}{5}\ln\left(\frac{59}{64}\right)}$ 1

$= 28.88929\ldots$

$\doteqdot 29$ seconds.

b) $y = \frac{1}{200} te^{-t}$

$\frac{dy}{dt} = \frac{1}{200}\left(t.-e^{-t} + e^{-t}.1\right)$

$= \frac{1}{200}(1-t)e^{-t}$ 1

c) i) when $t < 1$ $\frac{dA}{dt} > 0$ (since $(1-t) > 0$) 1

$\therefore$ A is increasing for $t < 1$

when $t > 1$ $\frac{dA}{dt} < 0$ (since $(1-t) < 0$) 1

$\therefore$ A is decreasing for $t > 1$

ii) When $t = 0$, $A = 0.0005$

$$A = \int \frac{dA}{dt}\,dt$$

$$= \int \frac{1}{200}(1-t)e^{-t}\,dt$$

$$= \frac{1}{200}te^{-t} + C \quad \text{(from part b))}$$ 1

when $t = 0$: $0.0005 = 0 + c$

$\therefore c = 0.0005$

$$\therefore A = \frac{1}{200}te^{-t} + 0.0005$$ 1

iii) from i) the maximum occurs when $t = 1$:

$$\therefore A_{max} = \frac{1}{200} \times 1 \times e^{-1} + 0.0005$$

$$= 0.002339397\ldots$$

$$\doteqdot 0.0023 \text{ (to 4 dec. pl.)}$$ 1

Question 6

a) i) tangents to a circle from an external point are equal.

$\therefore TX = XA$ (1)

So $\triangle TXA$ is isosceles and $\angle XTA = \angle XAT = \theta$

(base angles of isosceles $\triangle$)

ii) similarly, $\triangle AYS$ is isosceles with base angles $\angle YAS$ & $\angle YSA$ equal. (1)

But $\angle YAS = \angle TAX = \theta$ (vertically opposite)

So $\angle YSA = \theta$ & $\angle XTA = \theta$

But these are alternate (1)

So $TX \parallel YS$

b) Let $\angle ADE = \theta$ (1)

$\therefore \angle ABC = \theta$ (exterior angle cyclic quadrilateral) (1)

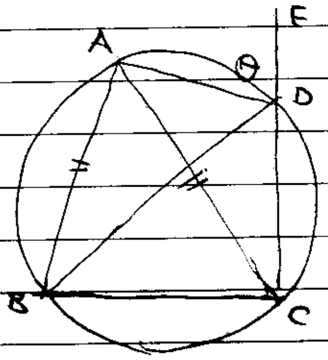
$\triangle ABC$ is isosceles ($AB = AC$, given)

$\therefore \angle ABC = \angle ACB = \theta$ (base angles isosceles $\triangle$) (1)

$\angle ACB = \angle ADB = \theta$ (angles in same segment) (1)

$\therefore \angle ADE = \angle ADB$ (both equal θ)

$\therefore DB$ bisects $\angle BDE$



c) i) $\angle PAF = \angle PBF = \alpha$ (angles in same segment) (1)

ii) $\angle ANB = \angle AMB = 90°$ (given)

$\therefore AB$ is a diameter of circle through A, N, M & B

(angle in a semicircle is a right $\angle$)

$\therefore$ A, N, M & B are concyclic. (1)

[OR angles in same segment / standing on chord AB....]

iii) $\angle NBM = \angle MAN$ (angles standing on same arc of circle ANMB) 1

$\angle MAN = \angle PAF$ (same angle)

$= \alpha$

$\therefore \angle NBM = \alpha$

iv) In Δ's HMB + FMB:

accept as proven above

1 { side: MB is common

angle: $\angle MBH = \angle MBF = \alpha$ ($\angle MBH$ is same as $\angle NBM$, $\angle MBF$ is same as $\angle PBF$)

1 { angle: $\angle HMB = \angle FMB = 90°$ (AM is altitude of ΔAPB)

$\therefore \Delta HMB \equiv \Delta FMB$ (AAS)

1 { $\therefore HM = FM$ (matching sides of congruent Δ's)

$\therefore$ M bisects HF.

(another option ... using $\tan\alpha = \ldots$)

Question 7

a) i) $m = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{\cancel{a}(p+q)\cancel{(p-q)}}{2\cancel{a}\cancel{(p-q)}}$ $(a \neq 0$ + $p - q \neq 0)$

$= \dfrac{p+q}{2}$ 1

$y - y_1 = m(x - x_1)$

$y - ap^2 = \dfrac{p+q}{2}(x - 2ap)$ 1

$\therefore y = \dfrac{p+q}{2}x - \cancel{\dfrac{2ap^2}{2}} - \dfrac{\cancel{2}apq}{\cancel{2}} + \cancel{ap^2}$

$\therefore y = \frac{1}{2}(p+q)x - apq$ as req'd.

ii) Focus $S(0, a)$ → satisfies equation if on chord:

$\therefore a = \frac{1}{2}(p+q) \times 0 - apq$ 1

$\therefore pq = \dfrac{a}{-a}$

$= -1$ as req'd.

iii) $PQ = PS + SQ$ (1)

$= \sqrt{(2ap)^2 + (ap^2 - a)^2} + \sqrt{(2aq)^2 + (aq^2 - a)^2}$

$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2} + \sqrt{4a^2q^2 + a^2q^4 - 2a^2q^2 + a^2}$

$= \sqrt{a^2(2p^2 + p^4 + 1)} + \sqrt{a^2(2q^2 + q^4 + 1)}$

$= \sqrt{a^2(p^2+1)^2} + \sqrt{a^2(q^2+1)^2}$

$= a(p^2+1) + a(q^2+1)$ (1)

$= a(p^2 + q^2 + 2)$

$= a\left(p^2 + 2 + \frac{1}{p^2}\right)$ (since $pq = -1$ & $q = -\frac{1}{p} \rightarrow q^2 = \frac{1}{p^2}$)

$= a\left(p + \frac{1}{p}\right)^2$ (1)

as required

b) i) $y = tx - t^2$

$A(x_a, 0)$: $0 = tx - t^2$

$\therefore x = t$

$\therefore A(t, 0)$ (1)

gradient OT:

$m_{OT} = \frac{t^2}{2t}$

$= \frac{t}{2}$

$\therefore m_{\perp OT} = -\frac{2}{t}$ (1)

$\therefore$ equation AP: $y - 0 = -\frac{2}{t}(x - t)$

$\therefore y = -\frac{2}{t}(x - t)$ as required

ii) equation OT : $y - 0 = \frac{t}{2}(x - 0)$ 1

$\therefore t = \frac{2y}{x}$ as required

iii) Locus of P → simultaneously solve AP & OT :

1 concept of simult. solution

$y = -\frac{2}{t}(x - t)$ — ①

$t = \frac{2y}{x}$ — ②

sub ② into ① :

$y = \frac{-\cancel{2}x}{\cancel{2}y}\left(x - \frac{2y}{x}\right)$ 1

$y^2 = -x^2 + 2y$

$x^2 + y^2 - 2y = 0$

$x^2 + y^2 - 2y + 1 = 1$ 1

$x^2 + (y-1)^2 = 1$

$\therefore$ locus is circle : centre $(0, 1)$, radius 1 [except the point $(0,0)$]