TRINITY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT

YEAR 12

MATHEMATICS EXTENSION 1

HALF-YEARLY EXAMINATION

ASSESSMENT TASK 3

Time Allowed - two hours
(Plus 5 minutes reading time)

Weighting - 30%

Morning Session

Friday 1st May 2009

General Instructions

- Write using blue or black pen.
- Begin each question on a new page.
- Place your number, class and teacher at the top of each page.
- Attempt ALL questions 1 - 7.
- All questions are of equal value.
- Mark values are shown at the side of each question.
- All necessary working should be shown in every question.
- Board-approved calculators may be used.
- Hand in a page for each question even if the question was not attempted.

Question 1. (12 Marks) **(Start this question on a NEW page)** **Marks**

(a) Find, correct to the nearest minute, the acute angle between the lines $2y-x-1=0$ and $y-3x+2=0$ **2**

(b) Find the exact value of $\sin\frac{\pi}{12}$. **2**

(c) Find the coordinates of a point P which divides the interval AB internally in the ratio of 1:3, if A is (-5, -1) and B is (3, 7). **2**

(d) Solve $\frac{1}{x(x-1)} \leq 0$ and graph the solution on a number line. **3**

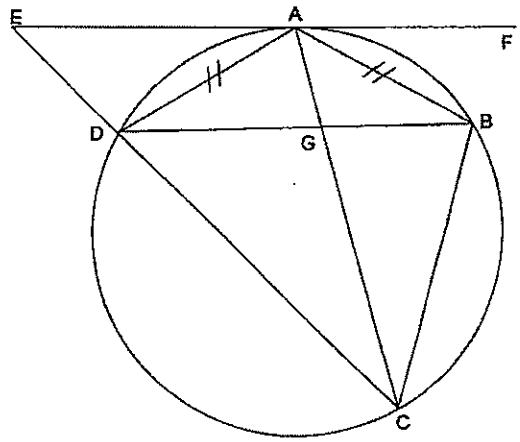
(e) State the domain and range of $f(x)=3\cos^{-1}2x$ and sketch the curve. **3**

Question 2. (12 Marks) **(Start this question on a NEW page)** **Marks**

(a) Find the general solution of the equation $2\sin\theta - \sqrt{3} = 0$. **2**

(b)

ABCD is a quadrilateral inscribed in a circle with $AB = AD$. The tangent at A meets CD produced at E.



Copy the diagram on to your paper.

(i) Prove $EF \parallel DB$. **2**

(ii) Prove AC bisects $\angle BCD$. **2**

(iii) Prove that $\Delta EAD \,|||\, \Delta ACB$. **2**

(c) Find the volume of the solid generated when the region between the the curve $y = x^2 - 4$ and the line $y = 2x - 4$ is rotated about the y- axis. **4**

Question 3. (12 Marks) **(Start this question on a NEW page)** **Marks**

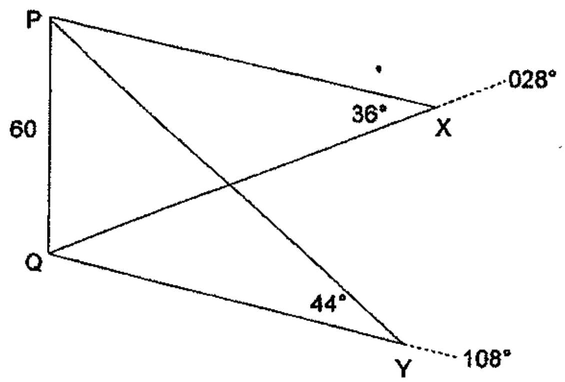
(a) Use the substitution $u = \sin 2x$ to evaluate $\int_0^{\frac{\pi}{4}} \sin^4 2x \cos 2x \, dx$ 3

(b) Using the principles of mathematical induction show that $\sum_{r=1}^{n} (3r-2) = \frac{n(3n-1)}{2}$ for all positive integers n. 3

(c) Solve $(w^2 - 2w)^2 - 2(w^2 - 2w) - 3 = 0$. 3

(d)

From an observation tower PQ of height 60 metres, two points X and Y at ground level have bearings 028° and 108°. From the top of the tower, the angles of depression of X and Y are 36° and 44° respectively.



(i) Prove that $XY^2 = 60^2\left(\cot^2 36^\circ + \cot^2 44^\circ - 2\cot 36^\circ \cot 44^\circ \cos 80^\circ\right)$ 2

(ii) Find XY to the nearest metre. 1

Question 4. (12 Marks) **(Start this question on a NEW page)** **Marks**

(a) A spherical bubble is expanding so that its volume is increasing at the constant rate $10\ cm^3$ / sec. What is the rate of increase of the radius when the surface area is $500\ cm^2$ / sec ? **2**

(b) (i) Find $\int \frac{1}{\sqrt{9-4x^2}}\,dx$ **2**

(ii) Evaluate $\int_0^{\pi} \frac{1}{2x+\pi}\,dx$ **2**

(c) Given the function $y = \log_e \sqrt{x^2-1}$, find:

(i) $\frac{dy}{dx}$ (ii) $\frac{d^2y}{dx^2}$ **3**

(d) Find the equation of the tangent to the curve $y = \sin^{-1}\frac{x}{2}$. at the point where $x = \sqrt{2}$. **3**

Question 5. (12 Marks) **(Start this question on a NEW page)** **Marks**

(a) (i) Express $\sqrt{3}\cos x - \sin x$ in the form of $R\cos(x+\alpha)$ **2**

(ii) Hence or otherwise solve $\sqrt{3}\cos x - \sin x = 1$ where $0 \le x \le 2\pi$. **2**

(b) Consider the function $f(x) = e^x - 1 - x$.

(i) Show that the minimum of $f(x)$ occurs at $x = 0$. **2**

(ii) Deduce that $f(x) \ge 0$ for all x. **1**

(iii) On the same set of axes, sketch $y = e^x - 1$ and $y = x$. **2**

(iv) Find the inverse of $g(x) = e^x - 1$. **1**

(v) State the domain of $g^{-1}(x)$. **1**

(vi) For what values of x is $\log_e(1+x) \le x$ **1**

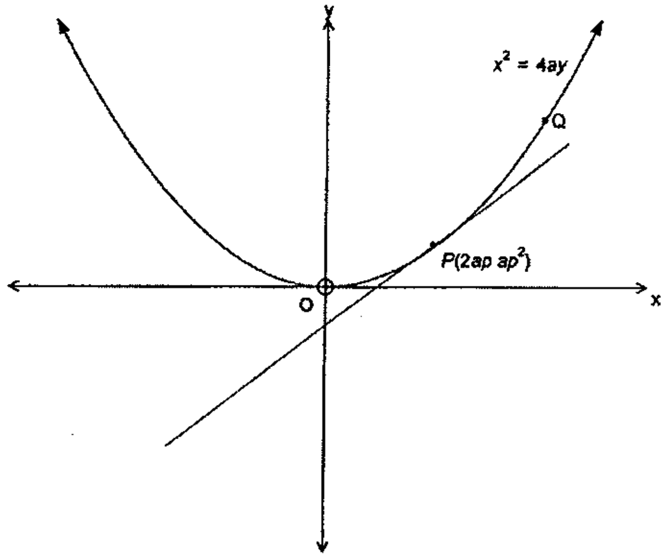
Question 6. (12 Marks) **(Start this question on a NEW page)** **Marks**

(a) In a game of Sic Bo, three regular six-sided dice are thrown once. Find the probability:

(i) of throwing three different numbers on the dice. **2**

(ii) that exactly two of the dice show the same number. **2**

(b) The variable point $P(2ap, ap^2)$ lies on $x^2 = 4ay$, and the chord OQ is drawn parallel to the tangent at P. The tangents at P and Q meet at R.



(i) Write down the equation of the chord OQ. **1**

(ii) Show that the coordinates of Q are $(4ap,\ 4ap^2)$. **2**

(iii) Find the equation of the tangent at Q. **1**

(iv) Show that R has the coordinates $(3ap,\ 2ap^2)$. **2**

(v) Find the Cartesian equation of the locus of R as P varies on the parabola. **2**

Question 7. (12 Marks) **(Start this question on a NEW page)** **Marks**

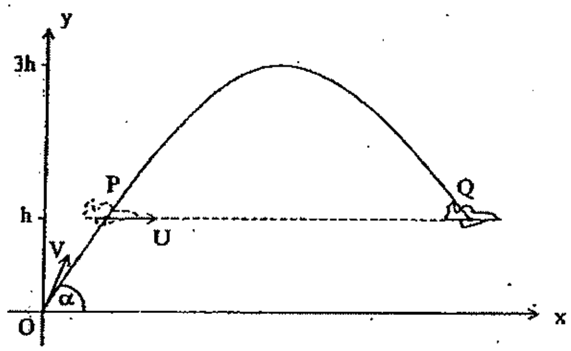
(a) The velocity $v\,ms^{-1}$ of a particle moving in a straight line is given by $v = 6 - 2x$, where x is the displacement in metres from the origin O and t is the time in seconds. Initially the particle is at rest at the origin.

(i) Find the acceleration at the origin. **2**

(ii) Show that $t = -\frac{1}{2}\log_e\left(1 - \frac{1}{3}x\right)$ **2**

(iii) Find x as a function of t. **2**

(b)



An enemy fighter plane is flying horizontally at height h metres with a speed $U ms^{-1}$. When it is at point P a ground rocket is fired towards it from the origin O with a speed $V ms^{-1}$ and angle of elevation α.

The rocket misses the plane, passing too late through point P. However, it goes on to reach a maximum height of $3h$ metres and then on its decent strikes the plane at Q. With the axes shown in the diagram above, you may assume that the position of the rocket is given by $x = Vt\cos\alpha$ and $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$ where t is the time in seconds after firing and g is the acceleration due to gravity.

(i) Show that initially the vertical component of the rocket's speed is $V\sin\alpha = \sqrt{6gh}$. **2**

(ii) If the rocket had not struck the fighter plane at Q, it would have returned to the x-axis at a distance d from O. Show that the horizontal component of the speed of the rocket is $V\cos\alpha = \frac{gd}{2\sqrt{6gh}}$. **2**

(iii) Show that the equation of the path of the rocket is $y = \frac{12hx}{d}\left(1 - \frac{x}{d}\right)$. **2**

TGS - Year 12 - Extension 1 Mathematics - May 2009

1. (a) $\tan\theta = \dfrac{3-\frac{1}{2}}{1+3(\frac{1}{2})}$ (M1)

$\theta = \tan^{-1}(1)$

$\theta = 45°$ (1)

(b) $\dfrac{1}{x(x-1)} \leq 0 \qquad x \neq 0,\ x \neq 1$

$x(x-1) \leq 0$ (M1)

$\therefore\ 0 < x < 1$

(1)

(c) $P\left[\dfrac{1(3)+3(-5)}{1+3}, \dfrac{1(7)+3(-1)}{1+3}\right]$ (M1)

$= (-3, 1)$ (1)

(d) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$ (M1)

$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$ (M1)

$= \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (1)

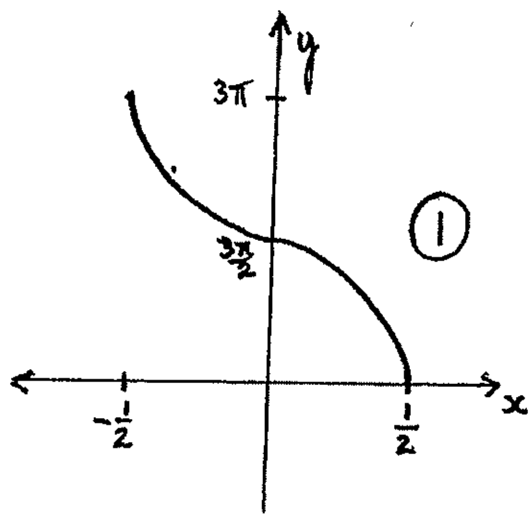
$= \dfrac{\sqrt{2}(\sqrt{3}-1)}{4}$

(e) D: $-1 \leq 2x \leq 1$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$ (1)

R: $0 \leq \cos^{-1}2x \leq \pi$

$0 \leq 3\cos^{-1}2x \leq 3\pi$ (1)



(1)

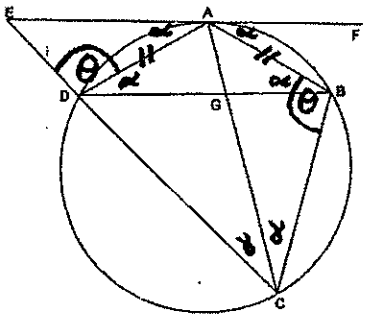
2.(a) $2\sin\theta - \sqrt{3} = 0$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}$ (1)

$\theta = n\pi + (-1)^n\left(\frac{\pi}{3}\right)$ (1)

(b)



i) $\angle ADG = \angle ABG$ Base ∠'s isos. △ =.

$\angle FAB = \angle ADG$ ∠ between tang. & chord = ∠ in alt. seg.

$\therefore \angle FAB = \angle ABG$ Equal ∠'s (1)

$EF \parallel DB$ Alt. ∠'s =. (1)

(ii) $\angle EAD = \angle ABG$ }

$\angle EAD = \angle ACD$ } ∠ between tang. & chord = ∠ in alt. seg. (1)

$\angle FAB = \angle BCA$ }

$\therefore \angle ACD = \angle BCA$ Equal angles (1)

iii) $\angle EAD = \angle BCA$ Equal angles

$\angle EDA = \angle ABC$ Ext. ∠ cyc. quad = int. opp. ∠. (1)

$\therefore \triangle EAD \parallel\!\!| \triangle ACB$ Two ∠'s △ =. (1)

c) $y = x^2 - 4 \quad y = 2x - 4$

$x^2 - 4 = 2x - 4$

$x(x-2) = 0$

$x = 0 \quad x = 2$

$y = -4 \quad y = 0$ (1)

$$V = \pi\int_{-4}^{0}(y+4)\,dy - \pi\int_{-4}^{0}\left(\frac{y+4}{2}\right)^2 dy \quad (1)$$

$$= \pi\int_{-4}^{0}\left[(y+4) - \left(\tfrac{1}{4}y^2 + 2y + 4\right)\right] dy$$

$$= \pi\int_{-4}^{0}\left(-y - \tfrac{1}{4}y^2\right) dy$$

$$= \pi\int_{0}^{-4}\left(y + \tfrac{1}{4}y^2\right) dy \quad (M1)$$

$$= \pi\left[\frac{y^2}{2} + \frac{y^3}{12}\right]_0^{-4}$$

$$= \pi\left(8 - \frac{16}{3}\right)$$

$$= \frac{8\pi}{3} \text{ units}^3 \quad (1)$$

3\. a) $\int_0^{\frac{\pi}{4}} \sin^4 2x \cos 2x \, dx$

$u = \sin 2x$

$du = 2\cos 2x \, dx$

when $x = 0$, $u = 0$; $x = \frac{\pi}{4}$, $u = 1$ (M1)

$= \frac{1}{2}\int_0^1 u^4 \, du$ (M1)

$= \frac{1}{2}\left[\frac{u^5}{5}\right]_0^1$

$= \frac{1}{10}$ (1)

(b) $1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}$.

let $n = 1$

L.H.S $= 1$ RHS $= \frac{1(3-1)}{2} = 1$ True for $n = 1$

Assume true for $n = k$.

$S_k \rightarrow 1 + 4 + 7 + \cdots + (3k-2) = \frac{k(3k-1)}{2}$

Prove true for $n = k+1$

$[1 + 4 + 7 + \cdots + 3k - 2] + [3(k+1) - 2]$

$= \frac{k(3k-1)}{2} + 3k + 1$

$= \frac{k(3k-1) + 6k + 2}{2}$ (M1)

$= \frac{3k^2 + 5k + 2}{2}$

$= \frac{(3k+2)(k+1)}{2} = $ R.H.S. (1)

If true for $n = k$, we have proven true for $n = k+1$; (1)
since true for $n = 1$, then true for $n = 1+1$, true for $n = 2, 3, 4, \ldots$
for all positive integers n.

(c) let $t = w^2 - 2w$

$t^2 - 2t - 3 = 0$ (M1)

$(t-3)(t+1) = 0$

$(w^2 - 2w - 3)(w^2 - 2w + 1) = 0$ (M1)

$(w-3)(w+1)(w-1)^2 = 0$

$\therefore w = \pm 1, 3$ (1)

(d) i) $QY = 60 \cot 44^\circ$

$QX = 60 \cot 36^\circ$

$\angle XQY = 108^\circ - 28^\circ = 80^\circ$ (M1)

In ΔXQY

$XY^2 = (60 \cot 44^\circ)^2 + (60 \cot 36^\circ)^2 - 2(60 \cot 36^\circ)(60 \cot 44^\circ)(\cos 80^\circ)$ (M1)

$= 60^2 \left[\cot^2 44^\circ + \cot^2 36^\circ - 2 \cot 36^\circ \cot 44^\circ \cos 80^\circ\right]$

ii) $XY = 94.33$

$= 94$ m (1) [must be correct to nearest metre]

4. a) $V = \frac{4}{3}\pi r^3$ $\qquad$ $S = 4\pi r^2$

$500 = 4\pi r^2$

$\frac{dV}{dt} = 10$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{10}{4\pi r^2}$ (M1)

$\frac{dr}{dt} = \frac{10}{500}$

$= \frac{1}{50}$ cm/sec. (1)

b)(i) $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$ (1)

$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + C$ (1)

ii) $\int_0^{\pi} \frac{1}{2x+\pi}\,dx$

$= \frac{1}{2}\left[\ln(2x+\pi)\right]_0^{\pi}$ (1)

$= \frac{1}{2}\left[\ln 3\pi - \ln \pi\right]$

$= \frac{1}{2}\ln 3$ (1)

(c) $y = \frac{1}{2}\ln(x^2-1)$

(i) $\frac{dy}{dx} = \frac{1}{2}\frac{2x}{(x^2-1)}$

$= \frac{x}{x^2-1}$ (1)

ii) $\frac{d^2y}{dx^2} = \frac{(x^2-1)(1) - (x)(2x)}{(x^2-1)^2}$ (M1)

$= \frac{-x^2-1}{(x^2-1)^2}$ (1)

(d) $y = \sin^{-1}\left(\frac{x}{2}\right)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\frac{x^2}{4}}} \cdot \frac{1}{2}$

$= \frac{1}{\sqrt{4-x^2}}$

$m_T = \frac{1}{\sqrt{2}}$ (1)

when $x = \sqrt{2}$

$y = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\pi}{4}$ (M1)

$(\sqrt{2}, \frac{\pi}{4})$

$y - \frac{\pi}{4} = \frac{1}{\sqrt{2}}(x - \sqrt{2})$

$y = \frac{1}{\sqrt{2}}x + \frac{\pi}{4} - 1$ (1)

5.(a)(i) $\sqrt{3}\cos x - \sin x = R[\cos x\cos\alpha - \sin x\sin\alpha]$

$R\cos\alpha = \sqrt{3}$ $R\sin\alpha = 1$

$R^2 = 3+1$ $\tan\alpha = \frac{1}{\sqrt{3}}$ (M1)

$\therefore R = 2$ $\therefore \alpha = \frac{\pi}{6}$

$\sqrt{3}\cos x - \sin x = 2\cos(x+\frac{\pi}{6})$ (1)

(ii) $2\cos(x+\frac{\pi}{6}) = 1$

$\cos(x+\frac{\pi}{6}) = \frac{1}{2}$ (M1)

$x+\frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$

$\therefore x = \frac{\pi}{6}, \frac{3\pi}{2}$ (1)

(b) (i) $f(x) = e^x - 1 - x$

$f'(x) = e^x - 1$

$f''(x) = e^x$

$f(0) = 1-1-0 = 0$ $(0,0)$

Stat. Points Max/Min occur when $f'(x)=0$

$e^x - 1 = 0$ (1)

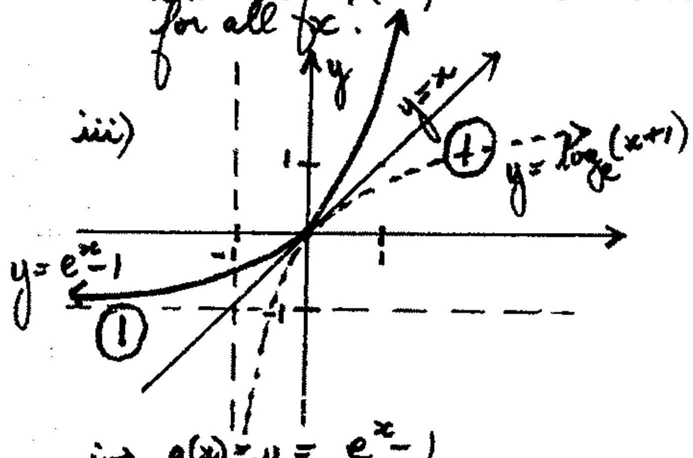
$e^x = 1$

$\therefore x = 0$

$f''(0) = 1$ $f''(0) > 0$ Minimum. (1)

(ii) Stat. pt. occurs at $(0,0)$ & the minimum value of $f(x) = 0$ when $x=0$. $\therefore f(x) \geq 0$ for all x. (1)

iii)



iv) $g(x) = y = e^x - 1$

$x = e^y - 1$

$x+1 = e^y$

$g^{-1}(x) = y = \log_e(x+1)$ (1)

(v) $D: x+1 > 0$

$x > -1$ (1)

(vi) Check the graphs of $y = \log_e(x+1)$ and $y = x$

$\log_e(x+1) \leq x$ for all $x > -1$ (1)

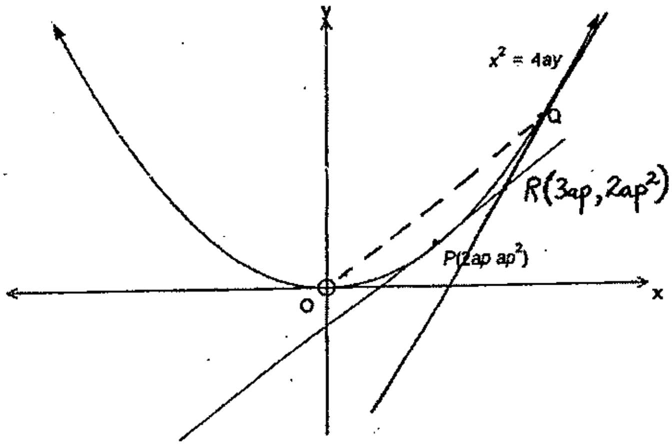
6. (a) i) $P(3 \text{ different nos.}) = 1 \times \frac{5}{6} \times \frac{4}{6}$ (M1)

$= \frac{5}{9}$ (1)

ii) $P(2 \text{ dice}^{\text{exactly}} \text{ have same number}) = \left[\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right] \times 3 \times 6$ (M1)

$= \frac{5}{12}$ (1)

b)



$x^2 = 4ay$

$y = \left(\frac{1}{4a}\right)x^2$

$y' = \left(\frac{1}{2a}\right)x$

(i) $m_{\text{tangent at } P} = p$

equation of chord $y = px$ (1)

ii) $px = y$ and $x^2 = 4ay$

$\therefore x^2 = 4apx$

$x(x - 4ap) = 0$ (M1)

$x = 0 \quad x = 4ap$

$y = p(4ap)$

$y = 4ap^2$ (1)

iii) $m_{\text{tangent at } Q} = \frac{4ap}{2a}$

$= 2p$

$y - 4ap^2 = 2p(x - 4ap)$ (M1)

$y - 4ap^2 = 2px - 8ap^2$

$y = 2px - 4ap^2$ (1)

iv) Equation of tangent at P

$y = px - ap^2$

$px - ap^2 = 2px - 4ap^2$ (M1)

$3ap^2 = px$

$3ap = x$

$y = p(3ap) - ap^2$

$= 2ap^2$ (1) $R(3ap, 2ap^2)$

(v) $x = 3ap$

$p = \frac{x}{3a}$

$y = 2a\left(\frac{x^2}{9a^2}\right)$

$y = \frac{2}{9a}x^2$ (1)

7(a) (i) $v = 6 - 2x$

$v^2 = 36 - 24x + 4x^2$ (M1)

$\frac{v^2}{2} = 18 - 12x + 2x^2$

$\ddot{x} = -12 + 4x$ when $x = 0$ $\ddot{x} = -12$ (1)

(ii) $\frac{dt}{dx} = \frac{1}{6-2x}$

$t = -\frac{1}{2}\ln(6-2x) + C$

$0 = -\frac{1}{2}\ln 6 + C$ $C = \frac{1}{2}\ln 6$ (M1)

$t = -\frac{1}{2}\left[\ln(6-2x) - \ln 6\right]$

$t = -\frac{1}{2}\ln\left(\frac{6-2x}{6}\right)$

$= -\frac{1}{2}\ln\left(1 - \frac{x}{3}\right)$ (1)

(iii) $-2t = \ln\left(1 - \frac{x}{3}\right)$

$1 - \frac{x}{3} = e^{-2t}$ (M1)

$\frac{x}{3} = 1 - e^{-2t}$

$x = 3(1 - e^{-2t})$ (1)

(b)(i) $\dot{y} = -gt + V\sin\alpha$

$t=0$ $\dot{y} = V\sin\alpha$

At max height, $\dot{y} = 0$, $y = 3h$

$0 = -gt + V\sin\alpha$

$t = \frac{V\sin\alpha}{g}$ (M1)

$y = -\frac{g}{2}\left(\frac{V\sin\alpha}{g}\right)^2 + V\sin\alpha\left(\frac{V\sin\alpha}{g}\right)$

$3h = -\frac{V^2\sin^2\alpha}{2g} + \frac{(2)V^2\sin^2\alpha}{(2)g}$ (M1)

$6gh = V^2\sin^2\alpha$

$\therefore V\sin\alpha = \sqrt{6gh}$

(ii) Let $y = 0$ $0 = t\left(-\frac{gt}{2} + V\sin\alpha\right)$

$t \neq 0$ $t = \frac{2V\sin\alpha}{g}$ (M1)

$d = V\cos\alpha\left(\frac{2V\sin\alpha}{g}\right)$

$V\cos\alpha = \frac{dg}{2\sqrt{6gh}}$ (1)

(iii) $x = V\cos\alpha\ t$

$t = \frac{x}{V\cos\alpha}$

$= \frac{2\sqrt{6gh}\,x}{gd}$ (M1)

$y = -\frac{g}{2}\left(\frac{2x\sqrt{6gh}}{gd}\right)^2 + \sqrt{6gh}\left(\frac{2\sqrt{6gh}\,x}{gd}\right)$

$= -\frac{12hx^2}{d^2} + \frac{12hx}{d}$

$= \frac{12hx}{d}\left(1 - \frac{x}{d}\right)$ (M1)