Trinity Grammar School

Mathematics Department

2013

HALF-YEARLY EXAMINATION

HSC ASSESSMENT TASK 3

Year 12

Mathematics Extension 1

Board of Studies Student Number
(Year 12 only)

Class Teacher:

.....

Name:
(Year 11 students)

.....

Do NOT write solutions on this question paper. Any working on this question paper will NOT be marked.

General Instructions

- Reading time – 5 minutes
- Writing time – 2 hours
- Write using black or blue pen
- Only approved calculators for this course are allowed in this task
- A table of Standard Integrals (if required) in this course is supplied
- Show all necessary working
- Write your Board of Studies Student Number (Year 12 HSC) or Name (Year 11) and your Class teacher on the question paper and on any answer sheets or booklets used to write your responses to the questions submitted
- If you do not attempt a question you must submit an answer sheet or writing booklet for that question clearly indicating **N/A** and your Student Number or Name.

- **HSC Assessment Weighting:** 30%

Total marks – 70

Section 1

10 marks

- Attemtpt all Questions
- Allow about **15** minutes for this section

Section II

60 marks

- Attempt all Questions
- Allow about **1** hours **45** minutes for this section

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Section I **10 marks**

- Shade the correct response on the answer sheet provided
- Each question is worth 1 mark

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1 If α, β and γ are the roots of $x^3 - 2x^2 + 3x - 1 = 0$, what is the value of $\alpha\beta + \beta\gamma + \alpha\gamma$?

(A) -3

(B) -2

(C) 2

(D) 3

2 Simplify $\dfrac{x^3-1}{x^2-1} \times \dfrac{x^2-4x-5}{4x^2+4x+4}$?

(A) $\dfrac{(x-5)}{4}$

(B) $\dfrac{(x-1)}{4}$

(C) $\dfrac{(x+1)}{4}$

(D) $\dfrac{(x^2+x+1)}{4}$

3 In which interval of $f(x) = \cos(x)$ is the inverse also a function?

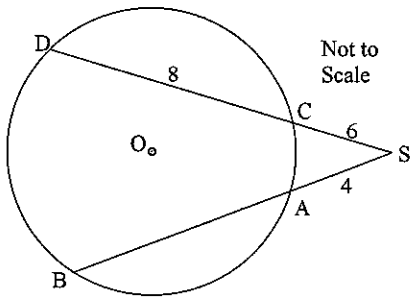
(A) $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

(B) $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$

(C) $0 \leq x \leq \pi$

(D) $\dfrac{\pi}{2} \leq x \leq \dfrac{3\pi}{2}$

4



In the circle, centre at O, SD intersects the circle at C and SB intersects the circle at A.

If SC = 6, CD = 8 and SA = 4, the length of AB is

(A) $\frac{16}{3}$ (B) 17 (C) 21 (D) $\frac{28}{3}$

5 Given $0 < A < \frac{\pi}{2}$, $\frac{\pi}{2} < B < \pi$, with $\sin A = \frac{12}{13}$ and $\cos B = -\frac{3}{5}$,

what is the value of $\cos(A - B)$?

(A) $\frac{33}{65}$ (B) $-\frac{33}{65}$ (C) $\frac{63}{65}$ (D) $-\frac{63}{65}$

6 If $x^{-1} + 1$ is divided by $x + 1$, the quotient equals

(A) 1 (B) $\frac{1}{x}$ (C) x (D) $-\frac{1}{x}$

7 The equation $x^2 + y^2 - 2x + 6y + 3 = 0$ is equivalent to

(A) $(x-1)^2 + (y+3)^2 = -3$

(B) $(x-1)^2 + (y+3)^2 = 7$

(C) $(x+1)^2 + (y+3)^2 = 7$

(D) $(x+1)^2 + (y+3)^2 = 10$

8 What is the derivative of $\cos^{-1}(4x)$?

(A) $\dfrac{1}{4\sqrt{1-16x^2}}$

(B) $\dfrac{-1}{4\sqrt{1-16x^2}}$

(C) $\dfrac{4}{\sqrt{1-16x^2}}$

(D) $\dfrac{-4}{\sqrt{1-16x^2}}$

9 The value of the expression $2\sum_{n=o}^{2}\left(n^2+2^n\right)$ is

(A) 12

(B) 22

(C) 24

(D) 26

10 If $\log x^2 - \log 2a = \log 3a$, then $\log x$ expressed in terms of $\log a$ is equivalent to

(A) $\frac{1}{2}\log 5a$

(B) $\frac{1}{2}\log 6 + \log a$

(C) $\log 6 + \log a$

(D) $\log 6 + 2\log a$

End of Section I

Section II 60 marks

- Begin each question in a new writing booklet or answer sheet
- In Questions 11-14 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Find the exact value of $\operatorname{cosec}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$ **1**

(b) Solve: $\frac{6^{3n}\times 9^{n+1}}{8^{n}}=1$. **2**

(c) State the domain and range of $y=3\sin^{-1}\left(\frac{x}{2}\right)$. **2**

(d) Solve $x-2<\frac{3}{x}$. **2**

(e) Differentiate $\log_e\left[\frac{1}{\sqrt{\sin x}}\right]$ with respect to x. **2**

(f) Sketch the graph of $y=\frac{2x^2}{x^2-4}$, showing all its important features, including the asymptotes. **3**

(g) The polynomial $P(x)=x^4-3x^3+ax^2+bx-6$ leaves a remainder of 8 when divided by $(x+1)$. If $(x-3)$ is a factor of $P(x)$, find the values of a and b. **3**

Question 12 (15 marks) **Begin a NEW answer booklet**

(a) Find the acute angle between the lines $2x-y+5=0$ and $x+3y=0$ giving your answer correct to the nearest minute. **3**

(b) Evaluate:

(i) $\int_0^{\frac{3}{2}} \frac{1}{4x^2+9}\,dx$ **3**

(ii) $\int_0^{\frac{3}{2}} \frac{x}{4x^2+9}\,dx$ **3**

(c) Find the equation of the normal to the curve $y=\sin^{-1}\left(\frac{x}{4}\right)$ at the point where $x=2\sqrt{3}$. **3**

(d) Use the substitution $u=x-2$ to evaluate $\int_2^6 x\sqrt{x-2}\,dx$. **3**

Question 13 (15 marks) **Begin a NEW answer booklet**

(a) Use mathematical induction to prove that $7^n + 2$ is divisible by 3 for $n \geq 1$. **3**

(b) A and B are the points (-5, 12) and (4, 9) respectively. P is the point which divides AB externally in the ratio of 5 : 2 . Find the coordinates of P . **3**

(c) Using $t = \tan\frac{x}{2}$, solve the trigonometric equation

$$\cos x - \sqrt{3}\sin x = -1 \quad \text{for} \quad 0 \leq x \leq 2\pi .$$ **3**

(d) The function $f(x) = e^x + x - 4$ has a zero near $x = 1$.

Use one application of Newton's method to obtain another approximation to this zero. Give your answer correct to two decimal places. **2**

(e) It is given that the rate of decrease of temperature of a body hotter than the surrounding air is proportional to the temperature difference. If A is the air temperature, and T is the temperature of the body after t minutes, then

$$\frac{dT}{dt} = -k(T - A)$$

(i) Show that if I is the initial temperature of the body, then the following function satisfies this condition:

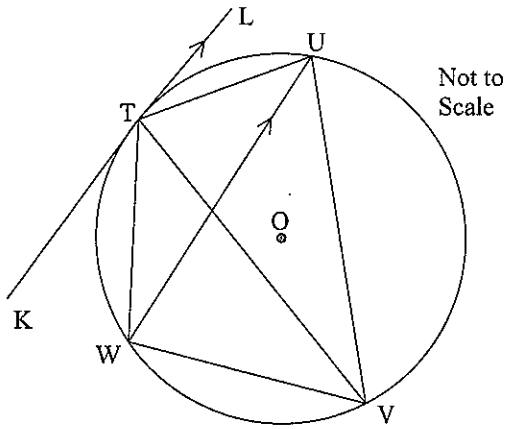
$$T = A + (I - A)\, e^{-kt}$$ **1**

(ii) An ingot of pig-iron, initially at a temperature of $500°C$, is allowed to cool in the open, where the temperature is $20°C$. If it cools to $200°C$ in five minutes, find the temperature of the ingot after ten minutes. **3**

Question 14 (15 marks) **Begin a NEW answer booklet**

(a) The height of a cone is 36 cm and remains constant, while the radius of the base is increasing at a rate of 6 cm/min. At what rate is the volume of the cone increasing when the radius is 24 cm? **2**

(b)



Points T, U, V and W lie on the circle with centre O.

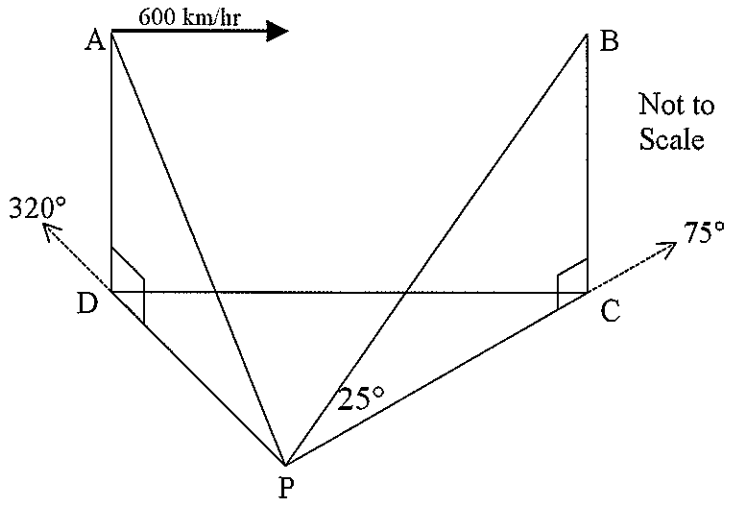
The line KL is tangent to the circle and parallel to the chord WU.

Copy the diagram on to your answer booklet.

Prove that TV bisects the angle WVU. **3**

Question 14 continues on page 10

(c)



The diagram shows a plane travelling at 600 km/hr at a constant altitude in the direction. shown. From an observation point P on the ground, the plane is sighted on a bearing of $320°$. One minute later, the bearing of the plane is $75°$ and its angle of elevation is $25°$.

(i) How far has the plane travelled between sightings? **1**

(ii) Show that the altitude h metres of the plane is given by

$$h = \frac{10\,000 \sin 50° \tan 25°}{\sin 65°}$$

and hence find h to the nearest metre. **3**

(iii) Find the angle of elevation of the plane from P when it was first sighted. **1**

(d) Consider the function $y = 1 + \cos 2x$.

(i) Sketch the curve $y = 1 + \cos 2x$ for $-\pi \leq x \leq \pi$. **2**

(ii) Find the volume generated when the area enclosed by the curve $y = 1 + \cos 2x$ and the x-axis for $-\pi \leq x \leq \pi$ is rotated about the x-axis. **3**

End of Section II

End of Half-Yearly Examination

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Trinity Grammar School
Mathematics Department
2013
Year 12 Mathematics Extension 1

HSC Assessment Task 3
Section I
Objective Response Answer Sheet

NSW Board of Studies Student Number
(Year 12 only)

Name: _____
(Year 11 only)

Class Teacher: _____

- Write your answers for **Section I** on this answer sheet.
- ***Shade completely only ONE answer for each question.***
- To change an answer, erase your previous mark completely and then record your new answer.

1	(A)	(B)	(C)	(D)
2	(A)	(B)	(C)	(D)
3	(A)	(B)	(C)	(D)
4	(A)	(B)	(C)	(D)
5	(A)	(B)	(C)	(D)
6	(A)	(B)	(C)	(D)
7	(A)	(B)	(C)	(D)
8	(A)	(B)	(C)	(D)
9	(A)	(B)	(C)	(D)
10	(A)	(B)	(C)	(D)

A table of Standard Integrals is printed on the reverse side of this answer sheet

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x \ , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

Trinity Grammar School
Mathematics Department
2013
Year 12 Mathematics Extension 1

HSC Assessment Task 3
Section I
Objective Response Answer Sheet

NSW Board of Studies Student Number
(Year 12 only)

Name: _____
(Year 11 only)

Class Teacher: _____

- Write your answers for **Section I** on this answer sheet.
- ***Shade completely only ONE answer for each question.***
- To change an answer, erase your previous mark completely and then record your new answer.

1	A	B	C	●
2	●	B	C	D
3	A	B	●	D
4	A	●	C	D
5	●	B	C	D
6	A	●	C	D
7	A	●	C	D
8	A	B	C	●
9	A	B	●	D
10	A	●	C	D

A table of Standard Integrals is printed on the reverse side of this answer sheet

2013 12MAX TASK 3 HALF-YEARLY EXAMINATION

Question 11.

a) $\operatorname{cosec}\left(\frac{\pi}{6}\right)$

$= \frac{1}{\left(\frac{1}{2}\right)}$

$= 2$ (1)

b) $\frac{2^{3n} \times 3^{3n} \times 3^{2n+2}}{2^{3n}} = 1$ (M1)

$3^{5n+2} = 3^{0}$

$\therefore 5n = -2$

$n = -\frac{2}{5}$ (1)

c) D: $-1 \leq \frac{x}{2} \leq 1$

$-2 \leq x \leq 2$ (1)

R: $-\frac{\pi}{2} \leq \frac{y}{3} \leq \frac{\pi}{2}$

$-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$ (1)

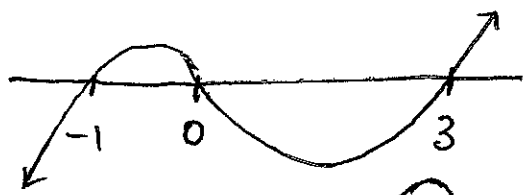
d) $x^2(x-2) < 3x$

$x^3 - 2x^2 - 3x < 0$

$x(x^2 - 2x - 3) < 0$

$x(x-3)(x+1) < 0$

$\therefore x < -1$ (1) OR $0 < x < 3$ (1)

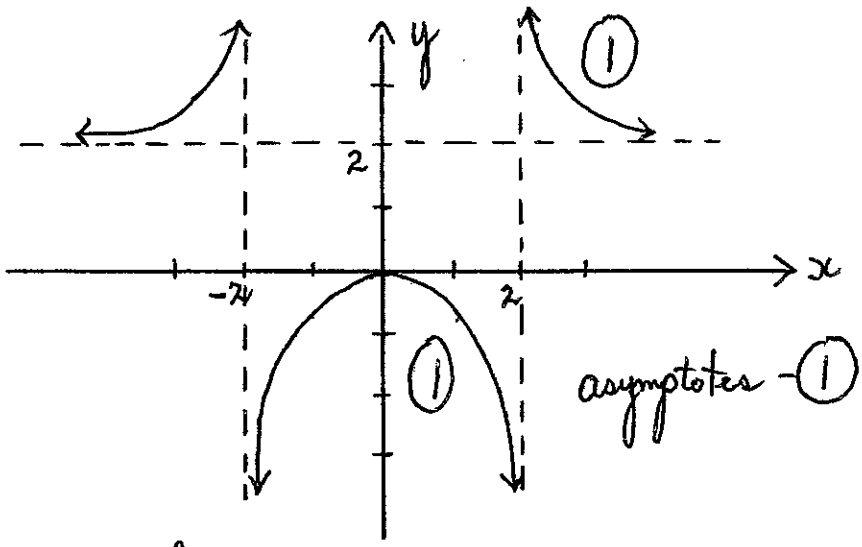


e) $\frac{d}{dx}\left[\log 1 - \frac{1}{2}\log \sin x\right]$ (M1)

$= 0 - \frac{1}{2} \cdot \frac{\cos x}{\sin x}$

$= -\frac{1}{2}\cot x$ (1)

f) $y = \dfrac{2x^2}{(x-2)(x+2)}$



g) $P(-1) = 1 + 3 + a - b - 6$

$8 = a - b - 2$

$a - b = 10 \quad \cdots (1)$

$P(3) = 81 - 81 + 9a + 3b - 6$

$0 = 3a + b - 2$

$3a + b = 2 \quad \cdots (2)$

(M1)

$\therefore 4a = 12$

$a = 3$ (1) $\quad b = -7$ (1)

Q12.

a) $m_1 = 2 \quad m_2 = -\frac{1}{3}$ (M1)

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{2 + \frac{1}{3}}{1 + (2)(-\frac{1}{3})}\right| \text{ (M1)}$$

$$\tan\theta = 7$$

$$\theta = 81°52' \text{ (1)}$$

b) i) $\int_0^{3/2} \frac{1}{4(x^2 + (\frac{3}{2})^2)}\,dx$ (M1)

$$= \frac{1}{4}\left[\frac{2}{3}\tan^{-1}\frac{2x}{3}\right]_0^{3/2}$$

$$= \frac{1}{6}\left[\tan^{-1}\frac{2}{3}\left(\frac{3}{2}\right) - \tan^{-1}0\right] \text{ (M1)}$$

$$= \frac{1}{6}\cdot\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{24} \text{ (1)}$$

ii) $\frac{1}{8}\int_0^{3/2} \frac{8x}{4x^2 + 9}\,dx$ (M1)

$$= \frac{1}{8}\left[\ln(4x^2 + 9)\right]_0^{3/2} \text{ (M1)}$$

$$= \frac{1}{8}\left[\ln 18 - \ln 9\right]$$

$$= \frac{1}{8}\ln 2 \text{ (1)}$$

c) $y = \sin^{-1}\left(\frac{x}{4}\right)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{16 - x^2}} \text{ (M1)}$$

at $x = 2\sqrt{3}$

$$m_T = \frac{1}{\sqrt{16 - 12}}$$

$$= \frac{1}{2}$$

$$\therefore m_N = -2$$

$$y = \sin^{-1}\left(\frac{2\sqrt{3}}{4}\right)$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{3}$$

$$y - \frac{\pi}{3} = -2\left(x - 2\sqrt{3}\right) \text{ (M1)}$$

$$y - \frac{\pi}{3} = -2x + 4\sqrt{3}$$

$$y = -2x + 4\sqrt{3} + \frac{\pi}{3} \quad \text{OR} \quad 2x + y - 4\sqrt{3} - \frac{\pi}{3} = 0 \text{ (1)}$$

d) $\int_2^6 x\sqrt{x-2}\,dx$

$u = x-2 \qquad x = u+2$

$du = (1)\,dx$

when $x=2 \quad u=0$

$x=6 \quad u=4$

$$= \int_0^4 (u+2)\sqrt{u}\,du$$

$$= \int_0^4 \left(u^{3/2} + 2u^{1/2}\right)du \quad (M1)$$

$$= \left[\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right]_0^4 \quad (M1)$$

$$= \left(\frac{2}{5}(32) + \frac{4}{3}(8)\right) - 0$$

$$= \frac{352}{15}$$

$$= 23\frac{7}{15} \quad (1)$$

Question 13.

a) $7^n + 2$ is divisible by 3 when $n \geq 1$

when $n = 1$

$7 + 2 = 9$ true for $n = 1$.

Assume true for $n = k$

$\therefore 7^k + 2 = 3M$ where k & M are integers.

$7^k = 3M - 2$ (M1)

Prove true for $n = k+1$

$7^{k+1} + 2$

$= 7 \times 7^k + 2$

$= 7(3M - 2) + 2$ (M1)

$= 21M - 14 + 2$

$= 3(7M - 4)$ which is divisible by 3 (M1)

If true for $n = k$, then true for $n = k+1$.

By Mathematical Induction true for all $n \geq 1$.

b) $P_x = \dfrac{5(4) - 2(-5)}{5 + (-2)}$ (M1) $= \dfrac{30}{3} = 10$

$P_y = \dfrac{5(9) - 2(12)}{5 + (-2)}$ (M1) $= \dfrac{21}{3} = 7$

$P(10, 7)$ (1)

c) $\left(\dfrac{1 - t^2}{1 + t^2}\right) - \sqrt{3}\left(\dfrac{2t}{1 + t^2}\right) = -1$

$1 - t^2 - 2\sqrt{3}t = -1 - t^2$ (M1)

$2 = 2\sqrt{3}t$

$t = \dfrac{1}{\sqrt{3}}$

$\tan \dfrac{x}{2} = \dfrac{1}{\sqrt{3}}$

$\dfrac{x}{2} = \dfrac{\pi}{6}, \dfrac{7\pi}{6}$

$\therefore x = \dfrac{\pi}{3}, \cancel{\dfrac{7\pi}{3}}$ (1)

Test $x = \pi$ $\cos \pi - \sqrt{3} \sin \pi$

$= -1 - 0$

$= -1$

$\therefore x = \dfrac{\pi}{3}, \pi$ (1)

d) $f(x) = e^x + x - 4$

$f'(x) = e^x + 1$

$x_1 = 1 - \dfrac{f(1)}{f'(1)}$

$= 1 - \dfrac{e-3}{e+1}$ (M1)

$= \dfrac{e+1-e+3}{e+1}$

$= \dfrac{4}{e+1}$

$= 1.08$ (1)

e)(i) $T = A + (I-A)e^{-kt}$

$T - A = (I-A)e^{-kt}$

$\dfrac{dT}{dt} = (I-A) \times -ke^{-kt}$

$= -k\left[(I-A)e^{-kt}\right]$ (M1)

$= -k(T-A)$

(ii) $t=0,\ I=500,\ A=20$

$t=5,\ T=200$

$200 = 20 + (500-20)e^{-5k}$ (M1)

$180 = 480e^{-5k}$

$e^{-5k} = 0.375$

$-5k = \ln(0.375)$

$k = 0.1961658506$ (1)

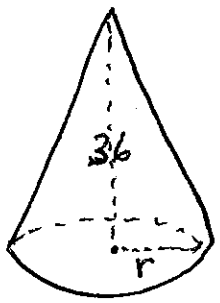
$t=10$

$T = 20 + 480e^{-10(0.1961658506)}$

$= 87.5^\circ C$ (1)

Question 14.

a)



$V = \frac{1}{3}\pi r^2 h$ $\quad h = 36\text{ cm}$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$ $\quad \frac{dr}{dt} = 6\text{ cm/min}$

$V = 12\pi r^2$

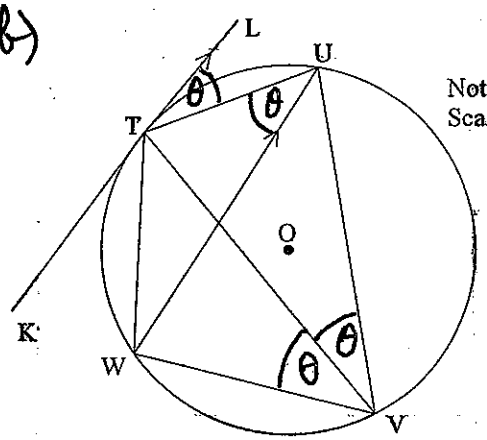
$\frac{dV}{dr} = 24\pi r$

$\therefore \frac{dV}{dt} = 24\pi(24) \times 6$ (M1)

$= 3456\pi\ \text{cm}^3/\text{min}$ (1)

$= 10857.3\ \text{cm}^3/\text{min}$

b)



$\angle LTU = \angle UVT = \theta$

($\angle$ between tangent & chord = $\angle$ in alt. segment) (M1)

$\angle LTU = \angle TUW$

(Alt. $\angle$'s =, $KL \parallel WU$) (M1)

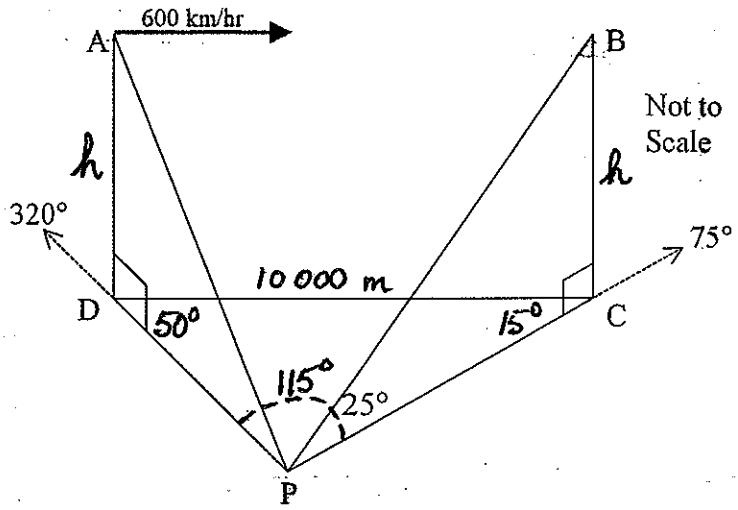
$\angle TUW = \angle TVW$

($\angle$'s in same segment =) (M1)

$\therefore \angle TVW = \angle TVU$

TV bisects $\angle WVU$.

c)



Bearings $(75° + 40°)$

$\angle DPC = 115°$

$\angle PDC = 50°$

$\therefore \angle PCD = 15°$

i) Distance $= 600 \times \frac{1}{60}$

$= 10$ km (1)

$= 10\,000$ m

ii) $\tan 25° = \frac{h}{PC}$ $\quad \sin 115° = \sin 65°$

$\therefore PC = \frac{h}{\tan 25°}$

In ΔDPC $\frac{10\,000}{\sin 115°} = \frac{PC}{\sin 50°}$ (M1)

$\frac{10\,000 \sin 50°}{\sin 65°} = PC = \frac{h}{\tan 25°}$ (M1)

$h = \frac{10\,000 \sin 50° \tan 25°}{\sin 65°}$

$= 3941$ m (1)

iii) $\frac{PD}{\sin 15°} = \frac{10\,000}{\sin 65°}$

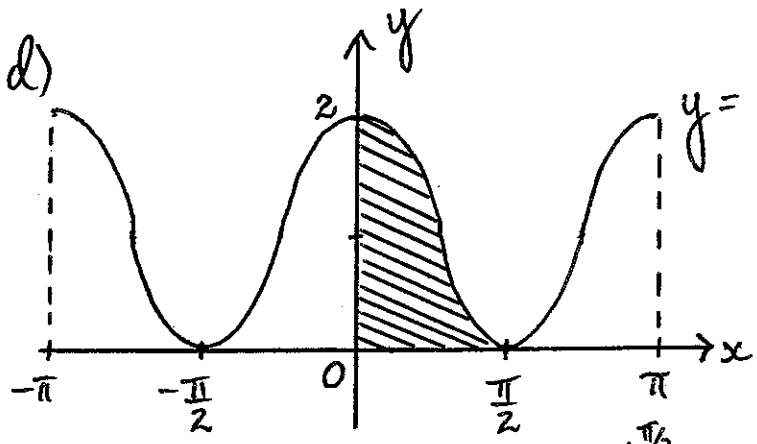
$PD = \frac{10\,000 \sin 15°}{\sin 65°} = 2856$

$\angle APD = \tan^{-1}\left[\frac{3941}{2856}\right]$

$= 54° 4'$

$= 54°$ (1)

d)



$$V = 4\times\pi\int_0^{\pi/2}(1+\cos 2x)^2\,dx$$

$$= 4\pi\int_0^{\pi/2}(1+2\cos 2x+\cos^2 2x)\,dx \quad (M1)$$

$$= 4\pi\int_0^{\pi/2}\left(1+2\cos 2x+\frac{1}{2}+\frac{1}{2}\cos 4x\right)dx$$

$$= 4\pi\left[\frac{3}{2}x+\sin 2x+\frac{1}{8}\sin 4x\right]_0^{\pi/2} \quad (M1)$$

$$= 4\pi\left[\left(\frac{3\pi}{4}+0+0\right)-(0)\right]$$

$$= 3\pi^2 \text{ units}^3 \quad (1)$$

$$\doteq 29.6 \text{ units}^3$$