ABBOTSLEIGH

YEAR 12 ASSESSMENT TEST

MATHEMATICS 4 UNIT

March 1998

Time Allowed: 1.5 hour

Question 1

(a) Using DeMoivre's theorem, or otherwise, find the square roots of $\sqrt{3}+i$

(b) Sketch on an Argand diagram, the locus of z defined by:

i) $\operatorname{Arg}\left(\frac{z-i}{z+1}\right)=0$ ii) $|z-(2+3i)|=25$

(c) A polynomial P(x) gives remainders of 2 and 6 when divided by x-1 and x+1 respectively. What must the remainder be when P(x) is divided by x^2-1 ?

(d) Sketch the ellipse $x^2+4y^2=16$, showing the foci and the directrices.

(e) State the domain and range of $y=2\sin^{-1}\left(\frac{3x-1}{2}\right)$ and make a neat sketch of the curve.

What is the slope of y when $x=\frac{1}{3}$?

Question 2 (Start a new page)

(a) Given that a monic polynomial Q(x) with rational coefficients has one root $1+2i$ and that 3 is a root of multiplicity 2, write the equation of the lowest possible degree of Q(x).

(b) Find the equation of the tangent to the ellipse $4x^2+9y^2=36$ at the point $\left(1,\frac{4\sqrt{2}}{3}\right)$.

Show that this tangent passes through the point $\left(6,\frac{1}{\sqrt{2}}\right)$. Hence find the points on the ellipse where the chord of contact from the point $\left(6,\frac{1}{\sqrt{2}}\right)$ intersects the ellipse.

hence

i) Factorise z^5-1 over the complex field

ii) Show that $\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=\frac{-1}{2}$

iii) If ω is one of the complex fifth roots of unity, show that $\omega+\omega^4$ and $\omega^2+\omega^3$ are the roots of the quadratic equation $x^2+x-1=0$

$\omega+\omega^4$ & $\omega^2+\omega^3$

Question 3 (Start a new page)

(a) Find the equation of the tangent to the rectangular hyperbola $xy=c^2$ at the point $\left(ct,\frac{c}{t}\right)$.

Also find where this tangent meets the x and y axis.

(b) Given that the polynomial $P(x)=x^5-6x^4+6x^3+20x^2-39x+18$ has two roots of multiplicity two, solve $P(x)=0$ $P(x)=x^5-6x^4+6x^3+20x^2-39x+18$

(c) i) State the domain of $y=(x-2)\sqrt{x-1}$

ii) Differentiate $y^2=(x-2)^2(x-1)$ and evaluate $\frac{dy}{dx}$ $y^2=(x-2)^2(x-1)$

(α) as $x\to 1$

(β) as $x\to -\infty$

(γ) when $x=2$

iii) Hence draw a neat sketch of the graph $y^2=(x-2)^2(x-1)$.

Question 4 (Start a new page)

(a) Resolve $\frac{x}{(x-3)(x+1)}$ into partial fractions, hence evaluate $\int_4^5\frac{2x}{(x-3)(x+1)}\,dx$

(b) i) Show that the condition for the line $y=mx+c$ to touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $c^2=a^2m^2-b^2$

ii) If P is the foot of the perpendicular from a focus S of the hyperbola to a variable tangent with slope m, by eliminating m from the equation of the tangent and perpendicular, find the locus of P.

(c) P:(3 secθ, 2 tanθ), and Q:(3 secθ, -2tanθ) are two points on a hyperbola of centre O.

i) Find the equation of the hyperbola. Make a sketch of the hyperbola, showing the foci, directrices and asymptotes.

ii) Plot the points P, Q for $\theta=\frac{\pi}{3}$

iii) If the normal to the hyperbola at P meets OQ in R, show that the locus of R is a hyperbola of centre O.

Abbotsleigh
Year 12 4u Maths
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Solutions

Q1 a) Let $z^2 = r^2(\text{cis}\,\theta)^2 = \sqrt{3}+i$

$r^2 = \sqrt{(\sqrt{3})^2+1} = 2 \quad \text{so } r = \sqrt{2}$

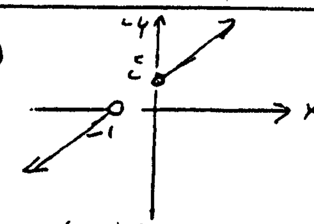
$(\text{cis}\,\theta)^2 = \text{cis}\,2\theta$

$\cos 2\theta = \sqrt{3}/2 \quad \sin 2\theta = \frac{1}{2}$

$\theta = \pi/12,\ -11\pi/12$ so sq roots

are $\sqrt{2}\,\text{cis}(\pi/12)$ and $\sqrt{2}\,\text{cis}\left(\frac{-11\pi}{12}\right)$

b) i)



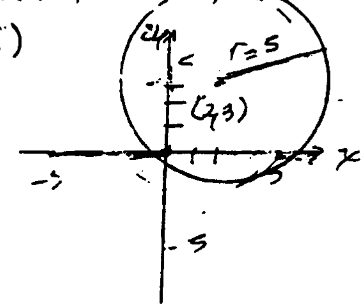
$\arg\left(\frac{z-i}{z+1}\right) = 0$

$\arg(z-i) - \arg(z+1) = 0$

$\arg(z-i) = \arg(z+1)$

$\therefore$ line: $y = x+1$ but not the interval from i to -1

ii)



$|z-(2+3i)| = 25$

$|(x-2)+(y-3)i| = 25$

$(x-2)^2+(y-3)^2 = 25$

$\therefore$ a circle centre $(2,3)$ radius 5

c) $P(x) = (x^2-1)Q(x) + ax+b$

$P(1) = a+b = 2$

$P(-1) = -a+b = 6$

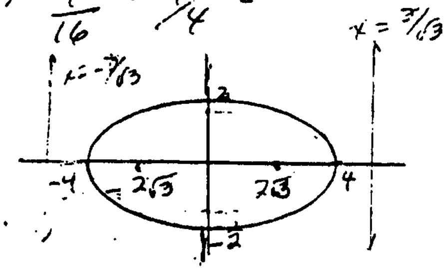
$2b = 8$

$b = 4$

$a = -2$

so remainder is $-2x+4$

d) $\frac{x^2}{16} + \frac{y^2}{4} = 1$



foci $(\pm ae, 0)$ $\quad ae = \sqrt{a^2-b^2}$
$= 2\sqrt{3}$

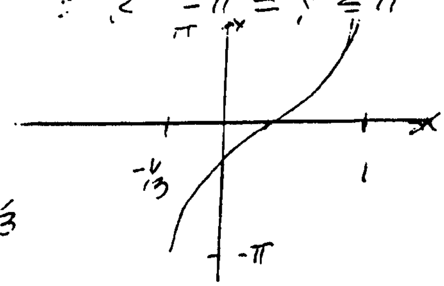
directrices $x = \pm a/e$
$= \pm \frac{8}{\sqrt{3}}$

e) D: $-1 \le \frac{3x-1}{2} \le 1$

$\therefore D: -\frac{1}{3} \le x \le 1$

R: $-\pi/2 \le \sin^{-1}\left(\frac{3x-1}{2}\right) \le \pi/2$

$\therefore 2 \quad -\pi \le y \le \pi$



$y' = \frac{2 \times \frac{3}{2}}{\sqrt{1-\left(\frac{3x-1}{2}\right)^2}}$

slope $= 3$. when $x = \frac{1}{3}$

Q2) a) $1-2i$ is also a root since coefficients are rational. so

equation $P(x) = (x-3)^2(x-1+2i)(x-1-2i)$

$= x^4 - 8x^3 + 26x^2 - 48x + 45$

or $P(x) = x^4 - \Sigma\alpha x^3 + \Sigma\alpha\beta x^2 - \Sigma\alpha\beta\gamma x + \alpha\beta\gamma\delta$

$= x^4 - 8x^3 + 26x^2 - 48x + 45$

where $\alpha, \beta = 3$ $\gamma = 1+2i$ $\delta = 1-2i$

b) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at $(1, \frac{4\sqrt2}{3})$

∴ tangent is $\frac{x}{9} + \frac{\sqrt2}{3}y = 1$ at $(1, \frac{4\sqrt2}{3})$

now $\frac{x}{9} + \frac{\sqrt2}{3}y = \frac23 + \frac13$ for $(6, \frac{1}{\sqrt2})$

$= 1$

∴ tangent contains $(6, \frac{1}{\sqrt2})$

Chord of contact is $\frac{6x}{9} + \frac{y}{4\sqrt2} = 1$

so $y = 4\sqrt2(1 - \frac23 x)$

for pts of intersection

$\frac{x^2}{9} + \frac{[4\sqrt2(1-\frac23 x)]^2}{4} = 1$

$x^2 + 72(1-\frac23 x)^2 = 9$ on simplifying

coeff of x^2 is 33, constant is 63

so $\alpha\beta = \frac{21}{11}$ $\alpha = 1$, so $\beta = \frac{21}{11}$

∴ pts are $(1, \frac{4\sqrt2}{3})$ and $(\frac{21}{11}, \frac{-12\sqrt2}{11})$

Q3) a) $x\,dy + y\,dx = 0$ $dy/dx = -y/x$

$= -1/t^2$ when $x = ct$

∴ T: $y - c/t = -\frac{1}{t^2}(x - ct)$

$t^2y + x = 2ct$ on x-axis $y = 0$

$(2ct, 0)$ and y-axis $x = 0$ $(0, \frac{2c}{t})$

b) $p'(x) = 5x^4 - 24x^3 + 18x^2 + 40x - 39$

$p'(1) = 5 - 24 + 18 + 40 - 39 = 0$

also $P(1) = 0$ so 1 is a double root

try factors of 18 say $P'(3) = 0$ & $P(3) = 0$

now product of roots $= -18$ so

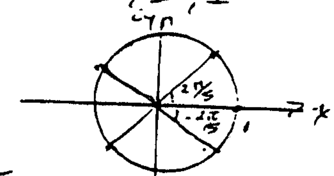
$P(2) = 0$ also note: only 5 solutions are possible ∴ 1, 1, 3, 3, 2

c) let $z = r\,\text{cis}\,\theta$

then $z^5 = \text{cis}\,5\theta = 1$

so roots are $\text{cis}\,\frac{2n\pi}{5}$

for $n = 0, \pm1, \pm2$



5 equally spaced roots o[n] circle $x^2 + y^2 = 1$

i) factors are $(z-1)(z-\text{ci}$

$(z - \text{cis}\frac{-2\pi}{5})(z - \frac{4\pi}{5})(z - \frac{-4\pi}{5})$

(ii) sum of roots $z^5 - 1 =$ is 0

so $1 + \text{cis}\pm\frac{2\pi}{5} + \text{cis}\pm\frac{4\pi}{5} = 0$

$\text{cis}\frac{2\pi}{5} + \text{cis}\frac{-2\pi}{5} = 2\cos\frac{2\pi}{5}$

$\text{cis}\frac{4\pi}{5} + \text{cis}\frac{-4\pi}{5} = 2\cos\frac{4\pi}{5}$

so $1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$

ie $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac12$

iii) $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

$(\omega + \omega^4) + (\omega^2 + \omega^3) = -1$

also $(\omega + \omega^4)(\omega^2 + \omega^3)$ (i) $=$

$\omega^3 + \omega^4 + \omega^6 + \omega^7 =$

$(\omega + \omega^4) + (\omega^2 + \omega^3)$ since

$\omega^6 = \omega^2$ & $\omega^7 = 1$

∴ product $= -1$

so quadratic is $x^2 + x + 1 =$

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Q3) i) domain $x \geq 1$

ii) $y^2 = (x-2)^2(x-1)$

$2y\,dy = [2(x-2)(x-1) + (x-2)^2]\,dx$

$\frac{dy}{dx} = \frac{(x-2)(3x-4)}{2y}$

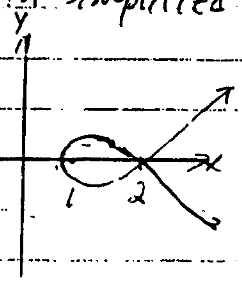
$= \frac{(x-2)(3x-4)}{\pm 2(x-2)\sqrt{x-1}}$

$= \pm \frac{(3x-4)}{2\sqrt{x-1}} \quad x \neq 1, 2$

α) as $x \to 1$ $\frac{dy}{dx} \to \pm\infty$

β) as $x \to +\infty$ $\frac{dy}{dx} \to \pm\infty$

γ) when $x = 2$ undetermined but ± 1 in simplified $\frac{dy}{dx}$



b) i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad y = mx + c$

$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$

$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(b^2+c^2)$

letting $\Delta = 0$

$(2a^2mc)^2 + 4(b^2 - a^2m^2)a^2(b^2+c^2)$

$4a^2b^2(-b^2 - c^2 + a^2m^2) = 0$

$c^2 = a^2m^2 - b^2$

ii) perpendicular at focus

$y - 0 = -\frac{1}{m}(x - ae)$

$x + my = ae$

tangent from i)

$y = mx + \sqrt{a^2m^2 - b^2}$

squaring both

$(x+my)^2 = a^2e^2$

$(y - mx)^2 = a^2m^2 - b^2$

Adding $(1+m^2)y^2 + (1+m^2)x^2 = a^2m^2 - b^2 + a^2e^2$

subst $b^2 = a^2(e^2 - 1)$

RHS $a^2(1+m^2)$

$\therefore x^2 + y^2 = a^2$ (divide by $1+m^2$)

a circle radius a and centre $(0,0)$

Q4) a) $\frac{x}{(x-3)(x+1)} = \frac{a}{x-3} + \frac{b}{x+1}$

$x = a(x+1) + b(x-3)$

if $x = -1$ $\quad -1 = -4b$

$\frac{1}{4} = b$

if $x = 3$ $\quad 3 = 4a$

$\frac{3}{4} = a$

$\therefore 2\int_4^5 \frac{x}{(x-3)(x+1)}\,dx = 2\int_4^5 \left[\frac{3}{4(x-3)} + \frac{1}{4(x+1)}\right]dx$

$= \frac{3}{2}\left[\ln(x-3)\right]_4^5 + \frac{1}{2}\left[\ln(x+1)\right]_4^5$

$= \frac{3}{2}(\ln 2 - \ln 1) + \frac{1}{2}(\ln 6 - \ln 5)$

$= \frac{3}{2}\ln 2 + \frac{1}{2}\ln\frac{6}{5}$

Q4) c) i) $P : (3\sec\theta, 2\tan\theta)$

$x = 3\sec\theta \qquad y = 2\tan\theta$

$\sec\theta = x/3 \qquad \tan\theta = y/2$

$\sec^2\theta - \tan^2\theta = 1$

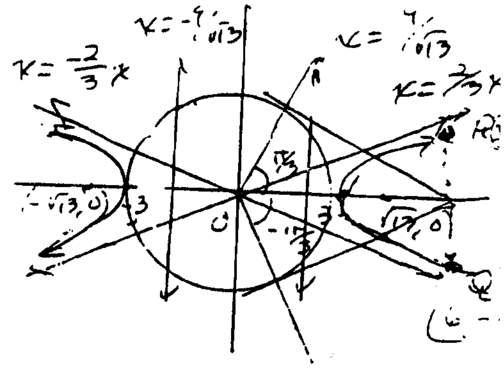
$\frac{x^2}{9} - \frac{y^2}{4} = 1$

$ae = \pm\sqrt{a^2+b^2}$ (foci)

$= \pm\sqrt{13}$

$\frac{a}{e} = \frac{9}{\sqrt{13}}$ (directrices)

$\pm\frac{b}{a}x = \pm\frac{2}{3}x$ (asymptotes)



ii) On diagram

iii) Normal: $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2+b^2$

equation OP $y = \frac{-2\tan\theta}{3\sec\theta}x$

Subst: $\frac{ax}{\sec\theta} + \frac{b(-2x)}{3\sec\theta} = a^2+b^2$

solving $\sec\theta = \frac{(3a-2b)x}{3(a^2+b^2)}$

Subst $x = \frac{3\sec\theta}{-2\tan\theta}$ into Normal

$\frac{3ay}{-2\tan\theta} + \frac{by}{\tan\theta} = a^2+b^2$

solving $\tan\theta = \frac{(3a-2b)y}{2(a^2+b^2)}$

$\sec^2\theta - \tan^2\theta = 1$

$\therefore \frac{(3a-2b)^2x^2}{9(a^2+b^2)^2} - \frac{(3a-2b)^2y^2}{4(a^2+b^2)} = 1$

which is a hyperbola centre $(0,0)$.